NON-GATHERABLE TRIPLES FOR CLASSICAL AFFINE ROOT SYSTEMS

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0. Introduction

This paper is a continuation of [CS] and the part of [C1] devoted to *non-gatherable triangle triples* in λ–sequences. The λ–sequences are the sequences of positive roots associated with reduced decompositions (words) in affine and nonaffine Weyl groups. The minimal *non-gatherable triangle triples*, NGT, $\{\alpha, \alpha+\beta, \beta\}$ is a λ–sequences with non-movable (under the Coxeter transformations) endpoints α, β such that $\alpha+\beta$ is a root and $|\alpha| = |\beta|$. Their nonaffine classification for $B_n, C_n(n \geq 3), D_n(n \geq 4)$ and for F_4, E_6 is the subject of [CS]; there are no NGT for nonaffine and affine A_n, B_2, C_2, G_2.

We describe all minimal NGT for the classical affine root systems based on their planar interpretation from [C3] and provide a universal general construction for arbitrary (reduced, irreducible) root systems. In principle, the latter can be used to obtain all such triples. *A posteriori*, only a significantly reduced version of our universal theorem is sufficient for the *classical* affine NGT, however, it can be more involved for the exceptional root systems.

The affine minimal NGT we construct are given in terms *almost dominant* weights (where one simple root can be disregarded in the definition of the dominant weights) and certain "small" elements from the nonaffine Weyl group. The weight itself generally is not sufficient to determine the corresponding minimal affine NGT uniquely.

For the classical affine root systems, the answer appeared very explicit. Combinatorially, it is given in terms of partitions of a type A subdiagram inside the initial nonaffine Dynkin diagram and (additionally) an increasing sequences of non-negative integers associated with such partitions.

Interestingly, all such minimal NGT can be naturally presented in terms of those of type B. It is not unexpected because the planar interpretation unifies all classical root systems in one construction. The passage to the other types from B is by using certain *parity corrections* directly related to the element s_0 of type B treated as an element of the *extended* affine Weyl groups of type C, D (in the twisted setting).

We note that there is a natural general (all root systems) procedure for producing candidates for *nonaffine* minimal NGT from the affine ones. For the exceptional root systems, it is justified only modulo certain technical assumptions. Employing it, we come to a somewhat

surprising construction of *nonaffine* minimal NGT in affine terms. It clarifies our nonaffine classification for the classical root systems and gives a promising approach to the exceptional nonaffine root systems, where the description of all minimal NGT is known but is technically involved.

The existence of NGT is a combinatorial obstacle for using the technique of intertwiners (see, e.g. [C1]) in the theory of irreducible representations of the affine and double affine Hecke algebras, complementary to the geometric approach from of [KL] and its double affine generalization. We mainly mean the constructive theory of such representations (where the intertwining elements are used to construct basic vectors).

The theory of affine and double affine algebras motivated our paper a great deal, but NGT are quite interesting in their own right. Gathering together the triangle triples using the Coxeter transformations seems an important question in the theory of reduced decompositions of Weyl groups, which is far from being simple. More generally, assuming that $\lambda(w)$ contains all positive roots of a certain root subsystem, can these roots be gathered using the Coxeter transformations?

Basic definitions. Let $R \in \mathbb{R}^n$ be a reduced irreducible root system or its affine extension, W the corresponding Weyl group. Then the λ–*set* is defined as $\lambda(w) = R_+ \cap w^{-1}(-R_+)$ for $w \in W$, where R_+ is the set of positive roots in R. It is well-known that w is uniquely determined by $\lambda(w)$; many properties of w and its reduced decompositions can be interpreted in terms of this set. The λ–*sequence* is the λ–set with the ordering of roots naturally induced by a given reduced decomposition.

The intrinsic description of such sets and sequences is mainly given in terms of the *triangle triples* $\{\beta, \gamma = \alpha+\beta, \alpha\}$. For instance, $\alpha, \beta \in \lambda(w)$ $\Rightarrow \alpha + \beta \in \lambda(w)$ and the latter root must appear between α and β if this set is treated as a sequence. This property is necessary but not sufficient; see [C1] for a comprehensive discussion.

We want to know when the sets of positive roots of rank two subsystems inside a given *sequence* $\lambda(w)$ can be *gathered* (made consecutive) using the Coxeter transformations in $\lambda(w)$. It is natural to allow the transformations only within the minimal segments containing these roots. This problem can be readily reduced to considering the *triangle triples* provided some special conditions on the lengths. The answer is always affirmative only for the root systems A_n, B_2, C_2, G_2 (and their

affine counterparts) or in the case when $|\alpha| \neq |\beta|$. Otherwise non-trivial NGT always exist.

The planar representation. For the root system A_n (nonaffine or affine), gathering the triples is simple. It readily results from the planar interpretation of the reduced decompositions and the corresponding λ–sequences in terms of $(n+1)$ lines in the two-dimensional plane (on the cylinder in the affine case).

Conceptually, this interpretation is a variant of the classical *geometric* approach to the reduced decompositions of $w \in W$ in terms of the lines (or pseudo-lines) that go from the main Weyl chamber to the chamber corresponding to w; see [B]. However, the planar description adds a lot to this general approach. It is a powerful tool, which dramatically simplifies dealing with combinatorial problems concerning the reduced decompositions.

The A_n–planar interpretation was extended in [C2] to other *classical* root systems and G_2, and then to their affine extensions in [C3]. Omitting G_2, it is given in terms of n lines in $\mathbb{R}^2$ with reflections in one *mirror* for the nonaffine B_n, C_n, D_n and two *mirrors* in the affine case. This approach is significantly developed in this paper; it can be used for quite a few problems beyond NGT.

We were able to use the planar interpretation to find *all* minimal non-gatherable triples, *minimal NGT*, for the affine root systems B, C, D. Algebraically, without such geometric support, it is an involved combinatorial problem. No planar (or similar) interpretation is known for $F_4, E_{6,7,8}$. Nonaffine minimal NGT can be classified using computers (see [CS] for F_4, E_6); the exceptional affine root systems will be considered in our further works.

Generally, the *admissibility* condition from [C1] is necessary and sufficient for the triple to be *gatherable*, which is formulated in terms of subsystems of R of types B_3, C_3 or D_4. This universal (but not very convenient to use) theorem can be now re-established for the classical root systems using the classification we give in this paper.

Relation to (double) affine Hecke algebras. The existence of NGT and some other features of similar nature are not present in the case of A. Generally, the theory of root systems is uniform at level of generators and relations of the corresponding Weyl (or braid) groups;

however the root systems behave quite differently when the "relations for Coxeter relations" are considered.

Presumably, the phenomenon of NGT is one of the major combinatorial obstacles for creating a universal theory of AHA-DAHA "highest vectors" generalizing Zelevinsky's *segments* in the A–case and based on the intertwining operators. This technique was fully developed only for affine and double affine Hecke algebras of type A_n and in some cases of small ranks.

The classification and explicit description of *semisimple* irreducible representations of AHA and DAHA is expected to be a natural application of this technique. The recent research (in progress) indicates that a thorough analysis of NGT is needed for this and similar projects.

The fact that all triples are gatherable in the case of A_n was the key in [C4] and quite a few further papers on the *quantum fusion procedure*. This procedure reflects the duality of AHA and DAHA of type A are the corresponding quantum groups and quantum toroidal algebras.

Quantum groups and Yangians certainly deserve special comments. In the case of GL, their irreducible representations can be described in terms of the so-called fusion procedure. The key object of the latter is the *transfer matrix*, a product of quantum R–matrices geometrically corresponding to a bunch of n parallel lines intersecting another bunch of m parallel lines.

Major parts of this big theory were extended to the R–matrices *with reflection* and the *twisted Yangians* (of reflection type). The the corresponding *transfer matrices* are associated with the following configurations. The n–bunch of lines intersects the m–bunch parallel to the mirror, then reflects in this mirror and then again intersects the m–bunch. There are interesting modifications here when D is considered. These configurations (when $n \geq 2$) are exactly those for the *non-affine* minimal NGT of type B, C. Recent research on the twisted Yangians [KN] indicates that it is not by chance and that minimal NGT may be of importance for this theory.

Expanding the theory of transfer matrices to the affine case is a natural challenge, including the *corner transfer matrices*, which are also related to our constructions. We hope that the classification of classical affine minimal NGT configurations will play its role in the (future) theory of twisted Yangians and Quantum groups of toroidal type.

1. Affine Weyl groups

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a Euclidean form (z, z') on $\mathbb{R}^n \ni z, z'$, W the *Weyl group* generated by the reflections s_α, R_+ the set of positive roots ($R_- = -R_+$) corresponding to fixed simple roots $\alpha_1, ..., \alpha_n$, Γ the Dynkin diagram with $\{\alpha_i, 1 \le i \le n\}$ as the vertices.

We will also use sometimes the dual roots (coroots) and the dual root system:

$$R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}.$$

The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the simple coroots $\alpha_i^\vee$. Replacing $\mathbb{Z}$ by $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \ge 0\}$ we obtain $Q_\pm, P_\pm$. Here and further see [B].

The form will be normalized by the condition $(\alpha, \alpha) = 2$ for *short* roots. When dealing with the classical root systems, the *most natural inner product* $(\, , \,)_\epsilon$ is the one making the ϵ_i in [B] orthonormal. It coincides with our $(\, , \,)$ for C and D; in the case of B, our form is $2(\, , \,)_\epsilon$. One has:

$$\nu_\alpha \stackrel{\mathbf{def}}{=\!=} (\alpha, \alpha)/2 \text{ can be either } 1, \text{ or } \{1, 2\}, \text{ or } \{1, 3\}.$$

This normalization leads to the inclusions $Q \subset Q^\vee, P \subset P^\vee$, where $P^\vee$ is defined to be generated by the fundamental coweights $\omega_i^\vee$.

Let $\vartheta \in R^\vee$ be the *maximal positive coroot*. Equivalently, it is *maximal positive short root* in R due to our choice of the normalization. All simple roots appear in its decomposition in R or $R^\vee$. Note that $2 \ge (\vartheta, \alpha^\vee) \ge 0$ for $\alpha > 0$, $(\vartheta, \alpha^\vee) = 2$ only for $\alpha = \vartheta$, and $s_\vartheta(\alpha) < 0$ if $(\vartheta, \alpha) > 0$.

1.1. **Affine roots.**

The vectors $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the *affine root system* $\widetilde{R} \supset R$ ($z \in \mathbb{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{\mathbf{def}}{=\!=} [-\vartheta, 1]$ to the simple roots for the *maximal short root* ϑ. The corresponding set $\widetilde{R}_+$ of positive roots coincides with $R_+ \cup \{[\alpha, \nu_\alpha j], \ \alpha \in R, \ j > 0\}$.

We will write $\widetilde{R} = \widetilde{A}_n, \widetilde{B}_n, \ldots, \widetilde{G}_2$ when dealing with classical root systems.

The root system $\widetilde{R}_+$ is called the *twisted* affine extension of R. The standard one from [B] is defined for maximal *long* root $\theta \in R_+$ and with omitting ν_α in the expression for the affine roots; the inner product is normalized by the condition $(\theta, \theta) = 2$. The transformation of our considerations to the non-twisted case is straightforward.

Any positive affine root $[\alpha, \nu_\alpha j]$ is a linear combinations with non-negative integral coefficients of $\{\alpha_i, 0 \le i \le n\}$. Indeed, it is well known that $[\alpha^\vee, j]$ is such combination in terms of $\{\alpha_i^\vee, 1 \le i \le n\}$ and $[-\vartheta, 1]$ for the system of affine *coroots*, that is $\widetilde{R}^\vee = \{[\alpha^\vee, j], \alpha \in R, j \in \mathbb{Z}\}$. Hence, $[-\alpha, \nu_\alpha j] = \nu_\alpha[-\alpha^\vee, j]$ has the required representation.

Note that the sum of the long roots is always long, the sum of two short roots can be a long root only if they are orthogonal to each other.

We complete the Dynkin diagram Γ of R by α_0 (by $-\vartheta$, to be more exact); it is called *affine Dynkin diagram* $\widetilde{\Gamma}$. One can obtain it from the completed (extended by zero) Dynkin diagram from [B] for the *dual system* $R^\vee$ by reversing all arrows.

The set of the indices of the images of α_0 by all the automorphisms of $\widetilde{\Gamma}$ will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O' = \{r \in O, r \neq 0\}$. The elements ω_r for $r \in O'$ are the so-called minuscule weights: $(\omega_r, \alpha^\vee) \le 1$ for $\alpha \in R_+$.

Given $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R}$, $b \in P$, let

$$s_{\widetilde{\alpha}}(\widetilde{z}) = \widetilde{z} - (z, \alpha^\vee)\widetilde{\alpha}, \quad b'(\widetilde{z}) = [z, \zeta - (z, b)] \tag{1.1}$$

for $\widetilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$.

The *affine Weyl group* $\widetilde{W}$ is generated by all $s_{\widetilde{\alpha}}$ (we write $\widetilde{W} = \langle s_{\widetilde{\alpha}}, \widetilde{\alpha} \in \widetilde{R}_+\rangle$). One can take the simple reflections $s_i = s_{\alpha_i}$ $(0 \le i \le n)$ as its generators and introduce the corresponding notion of the length. This group is the semidirect product $W \ltimes Q'$ of its subgroups $W = \langle s_\alpha, \alpha \in R_+\rangle$ and $Q' = \{a', a \in Q\}$, where

$$\alpha' = s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \text{ for } \alpha \in R. \tag{1.2}$$

The *extended Weyl group* $\widehat{W}$ generated by W and P' (instead of Q') is isomorphic to $W \ltimes P'$:

$$(wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \text{ for } w \in W, b \in P. \tag{1.3}$$

From now on, b and b', P and P' will be identified.

Note that the extended affine Weyl group in the standard (non-twisted case) is identified with the semidirect product $W \ltimes P^\vee$.

The action in $\mathbb{R}^{n+1}$ is dual to the *affine action* $\widehat{w}(\!(z)\!) \stackrel{\mathbf{def}}{=\!=} w(z+\xi b)$ in $\mathbb{R}^n \ni z$ for a free parameter ξ, where $\widehat{w} = wb$ and $w \in W, b \in P$. I.e., P acts via the translations in this definition. In more detail, let $(\,[z,t]\,,\, z'\,)_\xi \stackrel{\mathbf{def}}{=\!=} (z,z') + \xi t$ For $z, z' \in \mathbb{R}^n,\ t \in \mathbb{R}$ and $\widehat{w} = wb \in \widehat{W}$,

$$(\,\widehat{w}(z)\,,\, \widehat{w}(\!(z')\!)\,)_\xi \;=\; (\,z\,,\, z'\,)_\xi. \tag{1.4}$$

Note that $s_{[\alpha,j]}(\!(z)\!) = z - 2((z,\alpha) + j\xi)\alpha^\vee$

Given $b \in P_+$, let w_0^b be the longest element in the subgroup $W_0^b \subset W$ of the elements preserving b. This subgroup is generated by simple reflections. We set

$$u_b = w_0 w_0^b \in W,\ \pi_b = b(u_b)^{-1} \;\in\; \widehat{W},\ u_i = u_{\omega_i}, \pi_i = \pi_{\omega_i}, \tag{1.5}$$

where w_0 is the longest element in W, $1 \le i \le n$.

The elements $\pi_r \stackrel{\mathbf{def}}{=\!=} \pi_{\omega_r}, r \in O'$ and $\pi_0 = \mathrm{id}$ leave $\widetilde{\Gamma}$ invariant and form a group denoted by Π, which is isomorphic to P/Q by the natural projection $\{\omega_r \mapsto \pi_r\}$. As to $\{u_r\}$, they preserve the set $\{-\vartheta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$ distinguish the indices $r \in O'$. Moreover (see e.g., [C1]):

$$\widehat{W} = \Pi \ltimes \widetilde{W}, \quad \text{where} \quad \pi_r s_i \pi_r^{-1} = s_j \;\; \text{if} \;\; \pi_r(\alpha_i) = \alpha_j,\ 0 \le j \le n. \tag{1.6}$$

1.2. **The length.** Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ for $\pi_r \in \Pi, \widetilde{w} \in \widetilde{W}$, the length $l(\widehat{w})$ is by definition the length of the reduced decomposition $\widetilde{w} = s_{i_l} \ldots s_{i_2} s_{i_1}$ in terms of the simple reflections $s_i, 0 \le i \le n$.

The *length* can be also defined as the cardinality $|\lambda(\widehat{w})|$ of the $\lambda-$*set* of $\widehat{w}$:

$$\lambda(\widehat{w}) \stackrel{\mathbf{def}}{=\!=} \widetilde{R}_+ \cap \widehat{w}^{-1}(\widetilde{R}_-) = \{\widetilde{\alpha} \in \widetilde{R}_+,\ \widehat{w}(\widetilde{\alpha}) \in \widetilde{R}_-\},\ \widehat{w} \in \widehat{W}. \tag{1.7}$$

Note that $\lambda(\widehat{w})$ is closed with respect to positive linear combinations. More exactly, if $\widetilde{\alpha} = u\widetilde{\beta} + v\widetilde{\gamma} \in \widetilde{R}$ for rational $u, v > 0$, then $\widetilde{\alpha} \in \lambda(\widehat{w})$ if $\widetilde{\beta} \in \lambda(\widehat{w}) \ni \widetilde{\gamma}$. Vice versa, if $\lambda(\widehat{w}) \ni \widetilde{\alpha} = u\widetilde{\beta} + v\widetilde{\gamma}$ for $\widetilde{\beta}, \widetilde{\gamma} \in \widetilde{R}_+$ and rational $u, v > 0$, then either $\widetilde{\beta}$ or $\widetilde{\gamma}$ must belong to $\lambda(\widehat{w})$. Also,

$$\begin{aligned} &\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \lambda(\widehat{w}) \;\Rightarrow\; [\alpha, \nu_\alpha i] \in \lambda(\widehat{w}) \\ &\text{for all } 0 \le i < j \text{ where } i > 0 \text{ as } \alpha < 0\,. \end{aligned} \tag{1.8}$$

The coincidence with the previous definition is directly related to the equivalence of the following four claims:

$$(1.9)\qquad (a)\quad l(\widehat{w}\widehat{u}) = l(\widehat{w}) + l(\widehat{u}) \ \text{ for } \ \widehat{w}, \widehat{u} \in \widehat{W} \text{ (length formula)},$$

$$(1.10)\qquad (b)\quad \lambda(\widehat{w}\widehat{u}) = \lambda(\widehat{u}) \cup \widehat{u}^{-1}(\lambda(\widehat{w})) \text{ (cocycle relation)},$$

$$(1.11)\qquad (c)\quad \widehat{u}^{-1}(\lambda(\widehat{w})) \subset \widetilde{R}_+ \text{ (positivity condition)},$$

$$(1.12)\qquad (d)\quad \lambda(\widehat{u}) \subset \lambda_\nu(\widehat{w}) \text{ (embedding condition)}.$$

The key here is the following general relation:

$$(1.13)\qquad \lambda(\widehat{w}\widehat{u}) \ = \ \lambda(\widehat{u}) \,\widetilde{\cup}\, \widehat{u}^{-1}(\lambda(\widehat{w})) \ \text{ for any } \ \widehat{u}, \widehat{w},$$

where, by definition, the *reduced union* $\widetilde{\cup}$ is obtained from $\cap$ upon the cancelation of all pairs $\{\widetilde{\alpha}, -\widetilde{\alpha}\}$. In particular, (1.13) gives that

$$\lambda(\widehat{w}^{-1}) \ = \ -\widehat{w}(\lambda(\widehat{w})).$$

Applying (1.10) to the reduced decomposition $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_2} s_{i_1}$,

$$\lambda(\widehat{w}) = \{ \ \widetilde{\alpha}^l = \widetilde{w}^{-1} s_{i_l}(\alpha_{i_l}), \ \ldots, \ \widetilde{\alpha}^3 = s_{i_1} s_{i_2}(\alpha_{i_3}),$$

$$(1.14)\qquad \widetilde{\alpha}^2 = s_{i_1}(\alpha_{i_2}), \ \widetilde{\alpha}^1 = \alpha_{i_1} \ \}.$$

It demonstrates directly that the cardinality l of the set $\lambda(\widehat{w})$ equals $l(\widehat{w})$. Cf. [Hu],4.5.

Comment. It is worth mentioning that counterparts of the λ–sets can be introduced for $w = s_{i_l} \cdots s_{i_2} s_{i_1}$ in arbitrary Coxeter groups. Following [B] (Ch. IV, 1.4, Lemma 2), one can define

$$\Lambda(w) = \{ \ t_l = w^{-1} s_{i_l}(s_{i_l}), \ \ldots, \ t_3 = s_{i_1} s_{i_2}(s_{i_3}),$$

$$(1.15)\qquad t_2 = s_{i_1}(s_{i_2}), \ t_1 = s_{i_1} \ \},$$

where the action is by conjugation; $\Lambda(w) \subset W$.

The t–elements are (all) pairwise different if and only if the decomposition is reduced (a simple straight calculation; see [B]). Then this set does not depend on the choice of the reduced decomposition. It readily gives a proof of formula (1.14) by induction and establishes the equivalence of (a),(b) and (c).

Generally, the crystallographical case is significantly simpler than the case of abstract Coxeter groups; using the root systems dramatically simplifies theoretical and practical (via computers) analysis of the reduced decompositions. The positivity of roots, the alternative definition of the λ–sets from (1.7) and, more specifically, property (c) are (generally) missing in the theory of abstract Coxeter groups. $\square$

In this paper, we will mainly treat $\lambda(\widehat{w})$ as sequences, called λ–*sequences*; the roots in (1.14) are ordered naturally. The sequence structures of the same λ–set correspond to different choices of the reduced decompositions of $\widehat{w}$.

An arbitrary simple root $\alpha_i \in \lambda(\widehat{w})$ can be made the first in a certain λ–sequence. More generally:

$$\lambda(\widehat{w}) \;=\; \{\alpha > 0,\ l(\widehat{w}s_\alpha) \le l(\widehat{w})\}; \tag{1.16}$$

see [B] and [Hu],4.6, Exchange Condition.

The *sequence* $\lambda(\widetilde{w}) = \{\widetilde{\alpha}^l, \ldots, \widetilde{\alpha}^1\}$, where $l = l(\widetilde{w})$, determines $\widetilde{w} \in \widetilde{W}$ uniquely. Indeed,

$$\begin{aligned} &\alpha_{i_1} = \widetilde{\alpha}^1,\ \alpha_{i_2} = s^1(\widetilde{\alpha}^2), \ldots, \alpha_{i_p} = s^1 s^2 \cdots s^{p-1}(\widetilde{\alpha}^p), \ldots \\ &\alpha_{i_l} = s^1 s^2 \cdots s^{l-1}(\widetilde{\alpha}^l)\,, \quad \text{where} \\ &s^p = s_{\widetilde{\alpha}^p},\ \widehat{w} = s_{i_l} \cdots s_{i_1} = s^1 \cdots s^l. \end{aligned} \tag{1.17}$$

Notice the order of the reflections s^p in the decomposition of $\widetilde{w}$ is *inverse*. Moreover, $\lambda(\widehat{w})$ considered as an *unordered set* determines $\widehat{w}$ uniquely up to the left multiplication by the elements $\pi_r \in \Pi$.

The intrinsic definition of the λ–sequences is as follows.

(i) Assuming that $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} = \widetilde{\alpha} + \widetilde{\beta} \in \widetilde{R}_+$, if $\widetilde{\alpha}, \widetilde{\beta} \in \lambda$ then $\widetilde{\gamma} \in \lambda$ and $\widetilde{\gamma}$ appears between $\widetilde{\alpha}, \widetilde{\beta}$; if $\widetilde{\alpha} \notin \lambda$ then $\widetilde{\beta}$ belongs to λ and appears in λ before $\widetilde{\gamma}$.

(ii) If $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \lambda$ then $[\alpha, \nu_\alpha j'\,] \in \lambda$ as $j > j' > 0$ and it appears in λ before $\widetilde{\alpha}$.

(iii) If $\widetilde{\beta} \in \lambda$ and $\widetilde{\gamma} = \widetilde{\beta} - [\alpha, \nu_\alpha j] \in \widetilde{R}_+[-]$ for $\alpha \in R_+$, $j \ge 0$, then $\widetilde{\gamma} \in \lambda$ and it appears before $\widetilde{\beta}$.

If λ is treated as an unordered set, then it is in the form $\lambda = \lambda(\widehat{w})$ for some $\widehat{w} \in \widehat{W}$ if and only if $(i + ii + iii)$ are imposed without the claims concerning the ordering.

1.3. **Reduction modulo W.**

It generalizes the construction of the elements π_b for $b \in P_+$.

Proposition 1.1. *Given $b \in P$, there exists a unique decomposition $b = \pi_b u_b$, $u_b \in W$ satisfying one of the following equivalent conditions:*

(i) $\;l(\pi_b) + l(u_b) \;=\; l(b)$ *and $l(u_b)$ is the greatest possible,*

(ii) $\;\lambda(\pi_b) \cap R \;=\; \emptyset.$

The latter condition implies that $l(\pi_b) + l(w) = l(\pi_b w)$ *for any* $w \in W$. *Besides, the relation* $u_b(b) \stackrel{\mathbf{def}}{=\!=} b_- \in P_- = -P_+$ *holds, which, in its turn, determines* u_b *uniquely if one of the following equivalent conditions is imposed:*

(iii) $l(u_b)$ *is the smallest possible,*

(iv) if $\alpha \in \lambda(u_b)$ *then* $(\alpha, b) \neq 0$.

□

Condition (ii) readily gives a complete description of the λ–sets corresponding to the elements $\pi_P = \{\pi_b, b \in P\}$. The roots there must be from

$$\lambda \subset \widetilde{R}_+[-] \stackrel{\mathbf{def}}{=\!=} \{[\alpha,\, \nu_\alpha j], \alpha \in R_-, j > 0\}.$$

only See (1.8) and (1.19) below.

Setting $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R}_+$, one has:

$$\lambda(b) = \{\widetilde{\alpha},\ (b, \alpha^\vee) > j \geq 0 \ \text{ if } \ \alpha \in R_+, \quad (b, \alpha^\vee) \geq j > 0 \ \text{ if } \ \alpha \in R_-\}, \tag{1.18}$$

$$\lambda(\pi_b) = \{\widetilde{\alpha},\ \alpha \in R_-,\ (b_-, \alpha^\vee) > j > 0 \ \text{ if } \ u_b^{-1}(\alpha) \in R_+, \quad (b_-, \alpha^\vee) \geq j > 0 \ \text{ if } \ u_b^{-1}(\alpha) \in R_-\}, \tag{1.19}$$

$$\lambda(\pi_b^{-1}) = \{\widetilde{\alpha} \in \widetilde{R}_+,\ -(b, \alpha^\vee) > j \geq 0\}, \tag{1.20}$$

$$\lambda(u_b) \ = \ \{\alpha \in R_+, \quad (b, \alpha^\vee) > 0\}. \tag{1.21}$$

The element $b_- = u_b(b)$ is a unique element from P_- that belongs to the orbit $W(b)$. Thus the equality $c_- = b_-$ means that b, c belong to the same orbit. We will also use $b_+ \stackrel{\mathbf{def}}{=\!=} w_0(b_-)$, a unique element in $W(b) \cap P_+$. In terms of the elements π_b,

$$u_b \pi_b \ = \ b_-,\ \pi_b^{-1} u_b^{-1} \ = \ \varsigma(b_+),\ \varsigma(b) \stackrel{\mathbf{def}}{=\!=} -w_0(b).$$

Note that $l(\pi_b w) = l(\pi_b) + l(w)$ for all $b \in P,\ w \in W$. For instance,

$$l(b_- w) = l(b_-) + l(w),\ l(w b_+) = l(b_+) + l(w). \tag{1.22}$$

The definition of π_b and u_b is compatible with the one from (1.5) when $b \in P_+$. Namely,

$$u_b = w_0 w_0^b \in W,\ \pi_b = b(u_b)^{-1} \ \in \ \widehat{W} \ \text{ for } \ b \in P_+. \tag{1.23}$$

Recall that w_0^b is the longest element in the subgroup $W_0^b \subset W$ of the elements preserving b, w_0 is the longest element in W.

We will need below this construction extended to arbitrary $b \in P$ as follows. Let w_0^b be the longest element in the subgroup $W_0^b \subset W$, defined as the span of *simple* reflections $s_i (1 \le i \le n)$ preserving b. We set

$$\upsilon_b \stackrel{\mathbf{def}}{=\!=} w_0 w_0^b \in W, \ \varpi_b = b(\upsilon_b)^{-1} \ \in \ \widehat{W}. \tag{1.24}$$

For $b \in P_+$, the group W_0^b coincides with the complete centralizer of b in W; $\upsilon_b = u_b$ and $\varpi_b = \pi_b$. Note that $w_0 w_0^b = w_0^{b^\varsigma} w_0$ and

$$\varpi_b^{-1} = \varpi_{\varsigma(b)} \ \text{ for } \ \varsigma(b) = -w_0(b). \tag{1.25}$$

2. General theory of NGT

The transformations of the reduced decompositions in $\widetilde{W}$ are generated by the elementary ones, the *Coxeter transformations*, that are substitutions $(\cdots s_i s_j s_i) \mapsto (\cdots s_j s_i s_j)$ in reduced decompositions of the elements $\widetilde{w} \in \widetilde{W}$. The number of s–factors is $2, 3, 4, 6$ when α_i and α_j are connected by $0, 1, 2, 3$ laces in the affine or nonaffine Dynkin diagram. These transformations induce *reversing the order* of the corresponding segments (with $2, 3, 4, 6$ roots) of $\lambda(\widetilde{w})$ treated as a sequence. These segments can be naturally identified with the standard sequences of positive roots of type $A_1 \times A_1$, A_2, B_2 or G_2. The conjugations by $\pi_r \in \Pi$ will be applied too; they permute of the indices of the words from $\widetilde{W}$ (preserving the length).

2.1. **Admissibility condition.**

The theorem below is essentially from [C1]; it has application to the decomposition of the polynomial representation of DAHA and is important for the classification of semisimple representations of AHA and DAHA (in progress). We think that it clarifies why dealing with the intertwining operators for arbitrary root systems is significantly more difficult than in the A_n–case (where much is known).

Given a reduced decomposition of $\widehat{w} \in \widehat{W}$, let us assume that $\widetilde{\alpha} + \widetilde{\beta} = \widetilde{\gamma}$ for the roots $\ldots, \widetilde{\beta}, \ldots, \widetilde{\gamma}, \ldots, \widetilde{\alpha} \ldots$ in $\lambda(\widehat{w})$ ($\widetilde{\alpha}$ appears the first), where only the following combinations of their lengths are allowed in the $\widetilde{B}, \widetilde{C}, \widetilde{F}$ cases

$$\text{long} + \text{long} = \text{long} \ \ (B, F_4) \quad \text{or} \ \ \text{short} + \text{short} = \text{short} \ \ (C, F_4). \tag{2.1}$$

We call such $\{\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\alpha}\}$ a (triangle) *triple*.

Since we will use the Coxeter transformations only inside the segment $[\widetilde{\beta},\widetilde{\alpha}]\subset\lambda(w)$, from $\widetilde{\alpha}$ to $\widetilde{\beta}$, it suffices to assume that $\widetilde{\alpha}$ is a simple root. The root systems $\widetilde{A}_n,\widetilde{B}_2,\widetilde{C}_2,\widetilde{G}_2$ are excluded from the following theorem; there are no NGT in these cases.

Theorem 2.1. *The roots $\widetilde{\beta},\widetilde{\gamma},\widetilde{\alpha}$ from a triple are non-gatherable, i.e., cannot be made consecutive roots using the Coxeter transformations inside the segment $[\widetilde{\beta},\widetilde{\alpha}]\subset\lambda(\widehat{w})$ if and only if a root subsystem of type B_3, C_3 or D_4 exists such that its intersection with $\lambda(\widehat{w})$ constitutes the λ–set of a certain non-gatherable triple there.* □

The theorem can be readily reduced to considering the elements $\widehat{w}$ representing *minimal NGT*, i.e., such that the λ–sequence $\lambda(\widehat{w})$ begins with $\widetilde{\alpha}$ and ends with $\widetilde{\beta}$ and both roots (the endpoints) are non-movable with respect to the Coxeter transformations of $\widehat{w}$. Thus a minimal NGT is a pair, the triple $\{\widetilde{\beta},\widetilde{\gamma}=\widetilde{\alpha}+\widetilde{\beta},\widetilde{\alpha}\}$ and the element $\widehat{w}\in\widehat{W}$ that *represents* this triple. Since such triple is uniquely determined by $\widehat{w}$, we will constantly call $\widehat{w}$ minimal NGT too, somewhat abusing the terminology.

The classification of the classical affine minimal NGT gives this statement for the *classical* affine root systems. For the exceptional root systems, the first (universal, for all root systems) part of the paper can be used.

2.2. **Almost dominant weights.** Recall that we defined

$$\upsilon_b \stackrel{\mathbf{def}}{=\!=} w_0 w_0^b\in W,\ \varpi_b = b\,(\upsilon_b)^{-1}\ \in\ \widehat{W} \tag{2.2}$$

for an arbitrary $b\in P$ (not only for dominant ones). Given b, let us remove from the (nonaffine) Dynkin diagram Γ the vertices α_j such that $((\alpha_j,b)\neq 0$ and represent the output as a union of connected subdiagrams $\Gamma^{(m)}$. Thus:

$$\Gamma\ =\ \cup_m\Gamma^{(m)}\cup\{\alpha_j\}\text{ such that }\{\alpha_j\}=\{\alpha_j\mid(b,\alpha_j)\neq 0\}.$$

We will denote $\cup_m\Gamma^{(m)}$ by Γ^b; w_0^b is the product $\prod_m w_0^{(m)}$ for the longest elements in the Weyl groups $W^{(m)}$ defined for $\Gamma^{(m)}$. Note that $\Gamma^{\varsigma(b)}=\varsigma(\Gamma^b)$ for $\varsigma(b)=b^\varsigma=-w_0(b)$.

This definition will be mainly used for "almost dominant" b (when one simple root is omitted). Let us fix a nonaffine simple root α_k

$(1 \le k \le n)$. The weight $b \in P$ in the constructions below will be always assumed from

$$P_+^{(k)} \stackrel{\textbf{def}}{=\!=} \{\, a \in P \mid (a,\alpha_j) \ge 0 \ \text{ for } \ j \ne k,\ (a,\alpha_k) \le 0\,\}. \tag{2.3}$$

Notice that we allow here $(b,\alpha_k)=0$. Then the corresponding b will be dominant. In this case, there can be several choices for k; we pick one k such that $(b,\alpha_k)=0$, and construct $\dot{w}_0^b$ for $\Gamma_{\cdot}^b \stackrel{\textbf{def}}{=\!=} \Gamma^b\setminus\{\alpha_k\}$. Thus, $\dot{w}_0^b$ depends on the choice of k in this case. Accordingly, let $R_{+\cdot}^b$ be the set of positive roots of the root system associated with $\Gamma_{\cdot}^b$ considered as a subsystem of Γ. We will use below that $\lambda(\dot{w}_0^b) = R_{+\cdot}^b$.

Let $\upsilon_b^{\cdot} = w_0\dot{w}_0^b$ and $\varpi_b^{\cdot} = b(\upsilon_b^{\cdot})^{-1}$. For the sake of uniformity, the *dot*–notation will be used when $(b,\alpha_k)<0$; no *dot*–modifications of Γ^b, w_0^b, υ_b and ϖ_b are necessary in this case.

We need to extend the construction of $\varpi_b^{\cdot}$ even further by allowing certain reductions, the σ*–reductions*, of the elements $\upsilon_b^{\cdot}$. For $\sigma \in W$, we set

$$\upsilon_b^\sigma \ = \ \varsigma(\sigma)\upsilon_b^{\cdot} = w_0\sigma\dot{w}_0^b \ \in \ W,\ \ \varpi_b^\sigma \ = \ b\,(\upsilon_b^\sigma)^{-1} \ \in \ \widehat{W},$$

$$(\varpi_b^\sigma)^{-1} = \varpi_{\overline{b}}^{\overline{\sigma}} \ \text{ for } \ \overline{b} \stackrel{\textbf{def}}{=\!=} \varsigma(\sigma(b)),\ \overline{\sigma} \stackrel{\textbf{def}}{=\!=} \dot{w}_0^{\varsigma(b)}\,\varsigma(\sigma^{-1})\,\dot{w}_0^{\overline{b}}. \tag{2.4}$$

Accordingly,

$$\upsilon_{\overline{b}}^{\overline{\sigma}} = (\upsilon_b^\sigma)^{-1} = w_0\overline{\sigma}\dot{w}_0^{\overline{b}} \in W,\ \ \varpi_{\overline{b}}^{\overline{\sigma}} = \overline{b}\,(\upsilon_{\overline{b}}^{\overline{\sigma}})^{-1}.$$

The expression for $\overline{\sigma}$ is equivalent to the following relation:

$$\overline{\sigma}\dot{w}_0^{\overline{b}} \ = \ \dot{w}_0^{\varsigma(b)}\varsigma(\sigma^{-1}) \ = \ \varsigma(\sigma\dot{w}_0^{\varsigma(b)})^{-1}. \tag{2.5}$$

We note that $-\upsilon_b^\sigma(b) = \varsigma(\sigma(b)) = \overline{b}$, which readily gives that $\varsigma(\overline{\sigma}(\overline{b})) = b$. Thus, our *bar*–operation is involutive (by construction).

Recall that the definitions of $\dot{w}_0^b, \dot{w}_0^{\overline{b}}$ depend on the choice of $k, \overline{k}$ when $(\alpha_k, b) = 0$ and $(\alpha_{\overline{k}}, \overline{b}) = 0$ (they can be not unique such). The notation υ_b^σ automatically includes the *dot*–extension; if $\sigma = \mathrm{id}$ then $\upsilon_b^\sigma = \upsilon_b^{\cdot}$ and $\varpi_b^\sigma = \varpi_b^{\cdot}$.

Generally, the decompositions in (2.5) are not reduced. Let us address it.

Proposition 2.2. *For $b \in P_+^{(k)}$, we take $\sigma \in W$ such that $\overline{b} \stackrel{\textbf{def}}{=\!=} \varsigma(\sigma(b)) \in P_+^{(\overline{k})}$ for certain $1 \le \overline{k} \le n$. Then the following conditions*

$$(a):\quad l(\overline{\sigma}\dot{w}_0^{\overline{b}}) = l(\overline{\sigma}) + l(\dot{w}_0^{\overline{b}}), \qquad (b):\quad l(\sigma\dot{w}_0^b) = l(\sigma) + l(\dot{w}_0^b). \tag{2.6}$$

are correspondingly equivalent to:

$$(\widetilde{a}):\ \alpha_j \notin \lambda(\varpi_b^\sigma) \text{ for } 0 \neq j \neq \overline{k},\quad (\widetilde{b}):\ \alpha_j \notin \lambda(\varpi_{\overline{b}}^{\overline{\sigma}}) \text{ for } 0 \neq j \neq k.$$

Proof. It suffices to consider (a); the case of (b) is analogous (and formally follows from (a)). One has:

$$\varpi_b^\sigma \;=\; b(w_0\sigma\dot{w}_0^b)^{-1} = (\dot{w}_0^b\sigma^{-1}w_0)\cdot(-\overline{b}) = (w_0\overline{\sigma}\dot{w}_0^{\overline{b}})\cdot(-\overline{b}). \tag{2.7}$$

The set $\lambda(-\overline{b})$ contains $\alpha \in R_+$ if and only if $(-\overline{b},\alpha) > 0$ due to formula (1.18). Therefore, this set contains a simple nonaffine root only when $(-\overline{b},\alpha_{\overline{k}}) > 0$; then it can be only $\alpha_{\overline{k}}$. We use that $\overline{b} \in P_{+\cdot}^{(\overline{k})}$. Other nonaffine simple roots $\lambda(\varpi_b^\sigma)$ can come only from $\overline{b}(\lambda((w_0\overline{\sigma}\dot{w}_0^{\overline{b}}))$.

Conditions (a,b) are equivalent to:

$$(a'):\ \lambda(\overline{\sigma}\dot{w}_0^{\overline{b}}) \;=\; R_{+\cdot}^{\overline{b}} \cup \dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma})), \tag{2.8}$$

$$(b'):\ \lambda(\sigma\dot{w}_0^{b}) \;=\; R_{+\cdot}^{b} \cup \dot{w}_0^{b}(\lambda(\sigma)). \tag{2.9}$$

Recall that $\lambda(\dot{w}_0^{\overline{b}}) = R_{+\cdot}^{\overline{b}}$, where the latter is the set of all positive roots in the subsystem with simple (nonaffine) roots α_j such that $(\alpha_j,\overline{b}) = 0$ subject to the following *dot*–modification. If $(\alpha_{\overline{k}},\overline{b}) = 0$ (which is allowed), then $\alpha_{\overline{k}}$ must be excluded from $\{\alpha_j\}$.

We will use that $\lambda(w_0u) = R_+ \setminus \lambda(u)$ for any $u \in W$, which is obvious from the definition of the λ–sets.

Condition (a') is equivalent to the embedding $R_{+\cdot}^{\overline{b}} \subset \lambda(\overline{\sigma}\dot{w}_0^{\overline{b}})$. The set $\lambda(\overline{\sigma}\dot{w}_0^{\overline{b}})$ does not contain $R_{+\cdot}^{\overline{b}}$ if and only if at least one $\alpha_j \in R_{+\cdot}^{\overline{b}}$ is missing in the former set. Indeed, if all such α_j belong to this set then so do the roots that are their positive linear combinations.

If $\alpha_j \notin \lambda(\overline{\sigma}\dot{w}_0^{\overline{b}})$, then $\alpha_j \in \lambda(w_0\overline{\sigma}\dot{w}_0^{\overline{b}})$. Therefore $\alpha_j \in \varpi_b^\sigma$ due to (2.7) and because $(\alpha_j,\overline{b}) = 0$. It gives the required. $\square$

Let us express the embeddings

$$R_{+\cdot}^{\overline{b}} \subset \lambda(\overline{\sigma}\dot{w}_0^{\overline{b}}),\ \ R_{+\cdot}^{b} \subset \lambda(\overline{\sigma}\dot{w}_0^{b}),$$

using (2.5). Then (a') and (b') become equivalent correspondingly to

$$(a''):\quad \sigma(R_{+\cdot}^{b})\,\widetilde{\cup}\,\lambda(\sigma^{-1}) \;\supset\; \varsigma(R_{+\cdot}^{\overline{b}}) \ \text{ and} \tag{2.10}$$

$$(b''):\quad \overline{\sigma}(R_{+\cdot}^{\overline{b}})\,\widetilde{\cup}\,\lambda(\overline{\sigma}^{-1}) \;\supset\; \varsigma(R_{+\cdot}^{b}), \tag{2.11}$$

where $\widetilde{\cup}$ is the union where the pairs $\{\widetilde{\alpha}, -\widetilde{\alpha}\}$ are removed.

These conditions can be simplified if the following length conditions hold:

$$(\alpha):\ l(\dot{w}_0^{\overline{b}}\sigma^{\varsigma}) = l(\dot{w}_0^{\overline{b}}) + l(\sigma),\quad (\beta):\ l(\dot{w}_0^{b}\overline{\sigma}^{\varsigma}) = l(\dot{w}_0^{b}) + l(\overline{\sigma}). \tag{2.12}$$

Then (a'') and, correspondingly, (b'') become equivalent to

$$(a'''):\quad \sigma(R^b_{+\cdot}) \supset \varsigma(R^{\overline{b}}_{+\cdot}),\quad (b'''):\quad \overline{\sigma}(R^{\overline{b}}_{+\cdot}) \supset \varsigma(R^{b}_{+\cdot}). \tag{2.13}$$

For instance, let us check the equivalence of (a'') and (a'''). Using that $\lambda(\dot{w}_0^{\overline{b}}) = R^{\overline{b}}_{+\cdot}$ and that $\sigma(\lambda(\sigma)) = \lambda(\sigma^{-1})$,

$$\lambda(\sigma^{-1}) \cap \varsigma(R^{\overline{b}}_{+\cdot}) = \emptyset \iff (-\lambda(\sigma^{\varsigma})) \cap (\sigma^{\varsigma})^{-1}(R^{\overline{b}}_{+\cdot}) = \emptyset \iff (\alpha).$$

The requirements from (2.12) simply mean that the left-hand side or the right-hand side are reduced products in the following transformation of (2.5):

$$\dot{w}_0^{\overline{b}}\varsigma(\overline{\sigma}) \ = \ \sigma^{-1}\dot{w}_0^{\varsigma b}.$$

This holds in many examples of minimal NGT, though, generally, only for the left-hand side or only for the right-hand side (corresponding to (α) or (β)).

The setting from the next proposition simplifies the the construction significantly.

Proposition 2.3. *(i) Under the conditions $b \in P_+^{(k)}$ and $\overline{b} \in P_+^{(\overline{k})}$, let us assume that*

$$(a!b):\quad \sigma(\Gamma^b_{\cdot}) = \varsigma(\Gamma^{\overline{b}}_{\cdot})\ \ \textit{and}\ \ \sigma(\alpha_k) = \varsigma(\alpha_{\overline{k}}). \tag{2.14}$$

Then

$$\begin{aligned} \overline{\sigma} \ &= \ (\sigma^{\varsigma})^{-1},\ \ \overline{\sigma}(\overline{b}) \ = \ \varsigma(b),\ \ \dot{w}_0^{\overline{b}}\sigma^{\varsigma} \ = \ \sigma^{\varsigma}\dot{w}_0^{b^{\varsigma}}, \\ \lambda(\sigma^{-1}) \cap \varsigma(R^{\overline{b}}_{+\cdot}) \ &= \ \emptyset \ = \ \lambda(\overline{\sigma}^{-1}) \cap \varsigma(R^{b}_{+\cdot}). \end{aligned} \tag{2.15}$$

and the relations (a''', b''') from (2.13) are satisfied.

(ii) Moreover,

$$\lambda((\upsilon_b^{\sigma})^{-1}) = \lambda(w_0\dot{w}_0^{b^{\varsigma}}(\sigma^{\varsigma})^{-1}) = R_+ \setminus \big(R^{\overline{b}}_{+\cdot} \cup \lambda((\sigma^{\varsigma})^{-1})\big), \tag{2.16}$$

where $R^{\overline{b}}_{+\cdot} \cap \lambda((\sigma^{\varsigma})^{-1}) = \emptyset$, i.e., the union is disjoint. Equivalently,

$$l(\sigma) + l(\upsilon_b^{\sigma}) = l(\upsilon_b^{\cdot})\ \ \textit{for}\ \ \upsilon_b^{\cdot} = w_0\dot{w}_0^{b},\, \upsilon_b^{\sigma} = w_0\sigma\dot{w}_0^{b}.$$

The root $\beta = -w_0\sigma\dot{w}_0^b(\alpha_k)$ is positive, equivalently,

$$\alpha_k \in \lambda((\upsilon_b^{\sigma})^{-1}) = \lambda(\dot{w}_0^b\sigma w_0).$$

Proof. Relations (2.15) obviously follow from the assumptions from (2.14). Here $\lambda(\varsigma(\sigma)^{-1})\cap R^{\overline{b}}_{+\cdot}=\emptyset$ because $\varsigma(\sigma)^{-1}$ sends all simple roots from $R^{\overline{b}}_{+\cdot}$ to positive ones. Concerning (2.16),

$$
\begin{aligned}
\lambda(w_0\dot{w}_0^{b^\varsigma}) \;&=\; R_+\setminus\lambda(\dot{w}_0^{b^\varsigma}) \;=\; R_+\setminus R^{b^\varsigma}_{+\cdot},\\
\lambda(w_0\dot{w}_0^{b^\varsigma}(\sigma^\varsigma)^{-1}) \;&=\; R_+\setminus\big(R^{\overline{b}}_{+\cdot}\cup\lambda((\sigma^\varsigma)^{-1})\big).
\end{aligned}
\tag{2.17}
$$

It readily gives the desired length equality.

Let us check that $\beta=-w_0\sigma\dot{w}_0^b(\alpha_k)$ is positive. It suffices to assume that $(\beta,\overline{b})=(-\alpha_k,-b)=(\alpha_k,b)<0$. The positivity of β is equivalent to the positivity of $\varsigma(\sigma(\alpha_k))=\alpha_{\overline{k}}$. Indeed, $\dot{w}_0^b(\alpha_k)=\alpha_k+c$, where c is a linear combination of the simple roots from $\Gamma^b_{\cdot}$. Therefore,

$$
\beta \;=\; -w_0\sigma\dot{w}_0^b(\alpha_k) \;=\; \varsigma(\sigma(\alpha_k))+c',
$$

where c' is a linear combination of the simple roots from $\Gamma^{\overline{b}}_{\cdot}$. Since c' does not include $\alpha_{\overline{k}}$, the root β is positive. $\square$

If $b=0$ (the nonaffine case, which is allowed), then the condition $(a!b)$ implies that σ must be trivial. Therefore the σ–extension becomes the *dot*–extension in this case. Namely, $v^{\cdot}_{b=0}=u_k=u_{\omega_k}$ in the notations from (1.5).

In the case of trivial $\sigma=\mathrm{id}$,

$$
\overline{b}=\varsigma(b),\;\; \alpha_{\overline{k}}=\varsigma(\alpha_k)\;\;\text{ and }\;\; (\varpi^{\cdot}_b)^{-1}=\varpi^{\cdot}_{\varsigma(b)}\;\;\text{ and}
$$

$\beta=-w_0\dot{w}_0^b(\alpha_k)=\varsigma(\alpha_k)+\sum_{0\neq j\neq k}c_j\alpha_j^\varsigma$.

One can use the latter to analyze directly when the corresponding $\varpi_b=b(w_0\dot{w}_0^b)^{-1}$ is (represents) a minimal NGT, i.e., when $\gamma=\beta+\alpha_k^\varsigma$ is a root. Generally, it leads to explicit conditions for the coefficients c_j for simple roots α_j neighboring α_k in the Dynkin diagram.

Concerning the positivity of β for minimal NGT (which automatically results from the proposition), given an affine minimal NGT with $\beta<0$ of types $\widetilde{B},\widetilde{D}$, one can apply the transposition of the mirrors and make β positive (see below). Algebraically, this transformation corresponds to the symmetry $\alpha_0\leftrightarrow\alpha_n$. The case of $\widetilde{C}$ is analogous.

2.3. **Universal construction.** Let us address the root α_0. We use the notations and formulas from the previous section.

Proposition 2.4. *The following conditions for b,σ and $\overline{b},\overline{\sigma}$ are necessary and sufficient for $\alpha_0 \not\in \lambda(\varpi_b^\sigma)$ and $\alpha_0 \not\in \lambda(\varpi_{\overline{b}}^{\overline{\sigma}})$ correspondingly:*

$$\begin{aligned}
&(\overline{b},\vartheta)\le 0 \ \textit{or}\ \ (\overline{b},\vartheta)=1 \ \textit{if}\ \dot{w}_0^{\overline{b}}(\vartheta)\not\in\lambda(\overline{\sigma}),\\
&(b,\vartheta)\le 0 \ \textit{or}\ \ (b,\vartheta)=1 \ \textit{if}\ \dot{w}_0^{b}(\vartheta)\not\in\lambda(\sigma).
\end{aligned}\tag{2.18}$$

They imply that $b\not\in P_+$ and $\overline{b}\not\in P_+$ unless $b=0=\overline{b}$ or b and $\overline{b}$ are (both) minuscule.

Proof. First of all, we note that $\dot{w}_0^{\overline{b}}(\vartheta)\not\in\lambda(\overline{\sigma})$ is equivalent to $\upsilon_b^\sigma(\vartheta)<0$, which is sometimes easier to check. Using the definitions from (2.7) and general formula (1.13),

$$\begin{aligned}
\varpi_b^\sigma \ &= \ (\dot{w}_0^b\sigma^{-1}w_0)\cdot(-\overline{b})=(w_0\overline{\sigma}\dot{w}_0^{\overline{b}})\cdot(-\overline{b}),\\
\lambda(\varpi_b^\sigma) \ &= \ \overline{b}\Big(R_+\setminus\big(\dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma})\widetilde{\cup}R_{+\cdot}^{\overline{b}})\Big)\,\widetilde{\cup}\,\lambda(-\overline{b}),
\end{aligned}\tag{2.19}$$

where the modified union $\widetilde{\cup}$ includes, by definition, removing all possible pairs $\{\widetilde{\lambda},-\widetilde{\lambda}\}$.

The root $\alpha_0=[-\vartheta,1]$ can appear in $\lambda(\varpi_b^\sigma)$ only from $\lambda(-\overline{b})$ because all other roots there have positive nonaffine components. It exists in $\lambda(-\overline{b})$ if and only if $(-\overline{b},-\vartheta)=(\overline{b},\vartheta)\ge 1$; see (1.18). However, when it belongs to $\lambda(-\overline{b})$, it can be still canceled by $[\vartheta,-1]=\overline{b}(\vartheta)=[\,\vartheta\,,\,-(\overline{b},\vartheta)\,]$ under the following conditions

$$\vartheta\not\in\dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma}))\ \text{ and }\ (\overline{b},\vartheta)=1.$$

We use here that $\vartheta\not\in R_{+\cdot}^{\overline{b}}$ unless $b=0$.

Now let us assume that $\overline{b}\in P_+$ under the conditions for $\overline{b}$ from (2.18); the case of b is analogous. Then $(\vartheta,\overline{b})=(\vartheta,\varsigma(\overline{b}))>0$ unless $\overline{b}=0$. The conditions for $\overline{b}$ from (2.18) imply that $(\vartheta,\overline{b})=1$ and $\overline{b}$ is minuscule; so its λ–set is actually empty. $\square$

Note that if $\sigma=\mathrm{id}$, then $\overline{b}=\varsigma(b)$, $(\overline{b},\vartheta)=(b,\vartheta)$ and the conditions from (2.18) become a single inequality $(b,\vartheta)\le 1$. The next proposition addresses the occurrence of α_0 through the construction of the elements π_b from Proposition 1.1.

Proposition 2.5. *The elements $\widehat{w}\in\widehat{W}$ such that α_0 is a unique simple root in $\lambda(\widehat{w})$ and, moreover, there exists only one simple root in $\lambda(\widehat{w}^{-1})$*

are as follows:

$$(2.20)\quad \widehat{w}=\pi_b \ \ for\ \ b=-m\omega_i+\sum_{r\neq i}c_r\omega_j,\ 1\le i\le n, m\in\mathbb{Z}, c_r\in\mathbb{Z}_+.$$

Then the endpoints of $\lambda(\pi_b)$ are unique. Namely, this sequence begins with α_0 and ends either with α_0 for $m\le 0$ or with $[-\alpha_i,\nu_i m]$ if $m>0$. Such elements $\widehat{w}$ never represent minimal NGT.

Proof. The element $\widehat{w}$ with $\lambda(b)$ such that α_0 is its unique simple root must be in the form π_b; it is necessary and sufficient. Let us use (1.20):

$$\lambda(\pi_b^{-1})=\{[\alpha,\nu_\alpha j]\in\widetilde{R}_+,\ -(b,\alpha^\vee)>j\ge 0\}.$$

We see that (b,α_i) can be strictly negative only for one i such that $1\le i\le n$. This gives the representation from (2.20).

If $m\le 0$, then $2\alpha_0$ is not a root and the corresponding π_b is not a minimal NGT. If $m>0$, then the roots $\vartheta,\alpha_i,\vartheta+\alpha_i$ cannot (all) belong to a root subsystem of R of type A_2, since ϑ is the maximal short root. □

The following theorem is essentially a combination of the previous considerations. We mainly focus on the general problem of finding adequate presentations for $\widehat{w}\in\widehat{W}$ with non-movable endpoints in its $\lambda(\widehat{w})$. As an application, it provides a convenient universal tool (i.e., for all root systems) for managing the classification of the minimal NGT.

Here and below we will constantly use that if $\widehat{w}$ is with non-movable endpoints or is a minimal NGT then so is $\widehat{w}^{-1}$. Also,

$$(2.21)\quad \text{if } \widehat{w} \text{ is a minimal NGT, then so are } \pi_r\widehat{w} \text{ and } \widehat{w}\pi_r,\ \pi_r\in\Pi.$$

For instance, if $\widehat{w}=\pi_r\widetilde{w}\in\widehat{W}$ represents a minimal NGT, where $\widetilde{w}\in\widetilde{W}$, then $\widetilde{w}$ and $\pi_r(\widetilde{w})\pi_r^{-1}\in\widetilde{W}$ are minimal NGT too. The reduction $\widehat{w}\mapsto\widetilde{w}$ and conjugations by π_r can lead to quite non-trivial examples of minimal NGT (as words considered strictly inside $\widetilde{W}$) even if the initial $\widehat{w}$ is relatively simple.

Theorem 2.6. *(i) Under the conditions from (2.6) or, equivalently, (2.8), the set $\lambda(\varpi_b^\sigma)$ contains only one simple nonaffine root, $\alpha_{\overline{k}}$, the set $\lambda(\overline{\varpi}_b^\sigma)$ contains only α_k, where*

$$\overline{\varpi}_b^\sigma\ =\ (\varpi_b^\sigma)^{-1}\ =\ \varpi_{\overline{b}}^{\overline{\sigma}}.$$

Conditions (2.18) guarantee that $\lambda(\varpi_b^\sigma)\not\ni\alpha_0\notin\lambda(\overline{\varpi}_b^\sigma)$.

(ii) Equivalently, under the same conditions, the endpoints (the beginning and the end) of the sequence $\lambda(\varpi_b^\sigma)$ *are non-movable, i.e., do not depend on the choice of the reduced decomposition of* ϖ_b^σ*. Namely, this sequence begins with* $\alpha_{\overline{k}}$ *and ends with*

$$\widetilde{\beta}=-\varpi^{-1}(\alpha_k)=[\beta,p]\ \ \textit{for}\ \ \beta\stackrel{\mathbf{def}}{=\!=}-\upsilon(\alpha_k),\ p=-(b,\alpha_k)\geq 0, \tag{2.22}$$

where we set $\upsilon=\upsilon_b^\sigma,\ \varpi=\varpi_b^\sigma$*. It implies that* $b\notin P_+\not\ni\overline{b}$ *unless* $b=0=\overline{b}$*; thus, the dot-extension is actually needed only for zero* b*.*

(iii) All elements $\widehat{w}$ *with non-movable endpoints (both) are in the form* $\widehat{w}=\varpi_b^\sigma$ *for* $b\in P_+^{(k)}$ *and* $\overline{b}\in P_+^{(\overline{k})}$ *subject to the assumptions from (2.6) and (2.18), and also provided that* $\lambda(\widehat{w})\not\ni\alpha_0\notin\lambda(\widehat{w}^{-1})$*. The latter condition always holds if we assume that* $\gamma=\alpha_{\overline{k}}+\beta\in R$ *for* β *from (ii). In this case,* $\widetilde{\gamma}\stackrel{\mathbf{def}}{=\!=}\alpha_{\overline{k}}+\widetilde{\beta}=[\gamma,p]$ *belongs to* $\lambda(\varpi)$*. All minimal NGT can be obtained in this way.*

Proof. Part (i) is a combination of Proposition2.4 and Proposition 2.2. Formula (2.19) reads now:

$$\begin{aligned}&\lambda(\varpi_b^\sigma)\ =\ \overline{b}\Big(R_+\setminus\big(\dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma}))\cup R_{+\cdot}^{\overline{b}}\big)\Big)\,\widetilde{\cup}\,\lambda(-\overline{b}),\\ &\text{where } \dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma}))\cap R_{+\cdot}^{\overline{b}}\ =\ \emptyset.\end{aligned} \tag{2.23}$$

The only simple nonaffine root from $\lambda(\varpi_b^\sigma)$ can be $\alpha_{\overline{k}}$ due to formula (2.8). It comes from $\lambda(-\overline{b})$ when $p>0$. If $p=0$, then this root will remain in this set only if $\alpha_{\overline{k}}\notin\dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma}))\cup R_{+\cdot}^{\overline{b}}$. This case can be managed using a certain modification of (2.8). This can avoided because of the following argument. At least one simple root must be present in $\lambda(\varpi_b^\sigma)$; therefore it can be only $\alpha_{\overline{k}}$ since we excluded α_0. The case of $\lambda(\varpi_{\overline{b}}^{\overline{\sigma}})$ is entirely parallel.

Claim (ii) is a reformulation of (i). Note that, generally, conditions from (2.6), (2.8) and (2.18) make it possible to reduce any claims about the last roots (the ends) in the λ–sequences under consideration to the statements about the first roots (the beginnings). The interpretation of (ii) in terms of minimal NGT from (iii) is obvious.

Concerning the implication $b\in P_+\Longrightarrow b=0$, it was stated in Proposition 2.4. Recall that for dominant weights b, there is some flexibility with picking α_k and $\alpha_{\overline{k}}$ orthogonal to b and $\overline{b}$. Without the reference to Proposition 2.4, the required implication follows immediately from

the formula for $\widetilde{\beta}$. Indeed, since p is zero for $b \in P_+$, then $\widetilde{\beta} \in R_+$ and the triple $\{\widetilde{\beta}, \widetilde{\gamma}, \alpha_{\overline{k}}\}$ is nonaffine; as such, it can be represented only by a nonaffine element. The latter means that $b = 0$.

As for the completeness of our construction, we need to check only the fact that $\widehat{w}$ and $\widehat{w}^{-1}$ with non-movable ends can be obtained from *almost dominant* weights b and $\overline{b}$. The other conditions we imposed were actually necessary and sufficient.

Let us assume that $\widehat{w}$ satisfies the conditions from (iii) and set $\widehat{w} = wc$ for $c \in P, w \in W$. If $(\alpha_k, c) > 0 < (\alpha_{k'}, c)$ for $k \neq k'$, then $\lambda(c)$ contains two simple roots α_k and $\alpha_{k'}$. These roots cannot be canceled in $\lambda(wc) = c^{-1}(\lambda(w)) \widetilde{\cup} \lambda(c)$ because all roots from $c^{-1}(\lambda(w))$ have positive nonaffine components. This contradiction concludes (iii). $\square$

An example of type $\boldsymbol{D-E}$. Let us give a general construction applicable to all root systems of types $D_n (n \geq 4), E_6, E_7, E_8$. We take $\alpha_k = \alpha_4$ for $E_{6,7,8}$ and $\alpha_k = \alpha_{n-2}$ for D_n. The notation is from [B]; this root has three neighbors in Γ. Let $b = -\omega_k$, $\sigma = \mathrm{id}$. For D_n, one has $b = -(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{n-2}) \in P$. Then the element $\overline{b} = \varsigma(b)$ coincides with b. Thus, $k = \overline{k}$ ($k = 4$ for D_4, E). Since $(b, \theta) < 0$, the conditions from (2.18) are obviously satisfied.

One has: $\beta = -w_0 w_0^b(\alpha_k) = \sum_{j=1}^{n} \alpha_j$. Here $w_0 w_0^b$ sends

$$\epsilon_j \mapsto -\epsilon_{l-1-j} \ \text{ for } \ j \leq l-2, \ \epsilon_{n-1} \mapsto \epsilon_{n-1}, \ \epsilon_n \mapsto -\epsilon_n.$$

The relation $\alpha_k + \beta \in R$ is not immediately clear but readily follows from the tables of [B]. Thus, $\varpi = b w_0^b w_0$ represents the minimal NGT $\{[1, \beta], [1, \alpha_k + \beta], \alpha_k\}$.

From the viewpoint of Theorem 4.2 below, the geometric classification theorem, this example gives the simplest possible minimal NGT for D_n. The corresponding configuration has two bottom horizontal lines (the smallest possible number); the other lines form a bunch of "parallel lines" with one reflection in the top mirror and 2 in the bottom one.

If n is odd here, then $\omega_k \notin Q$ and a *parity correction* is needed if we want to reduce ϖ to $\widetilde{W}$. Namely, $\varpi' = \pi_1 \varpi \in \widetilde{W}$ and $\varpi'' = \varpi \pi_1 = (\varpi')^{-1} \in \widetilde{W}$ are minimal NGT for $\pi_1 = \pi_{\omega_1} = \epsilon_1 s_{\epsilon_1} s_{\epsilon_n}$. The corresponding weights b in the decomposition $\widetilde{w} = bw, b \in Q$ are

$$b' = 2\epsilon_1 - (\epsilon_2 + \ldots + \epsilon_{n-2}), \ b'' = -(\epsilon_1 + \ldots + \epsilon_{n-3} + 2\epsilon_{n-2}). \tag{2.24}$$

Note that the elements $\varpi_{b'}$ and $\varpi_{b''}$, which are minimal NGT too, are different from ϖ' and ϖ'', although the corresponding weights coincide. $\square$

The theorem reduces the classification of all elements $\widehat{w}$ with non-movable endpoints of their λ–sequences to *finitely many* verifications. The analysis is involved for the exceptional root systems. However, we expect that the classification of minimal NGT (a subclass of all $\widehat{w}$ with non-movable λ–endpoints) is not that ramified. For instance, the description of all minimal NGT satisfying Proposition 2.3 seems quite doable (although this setting is not sufficient for all of them). It is similar to the verifications in the example above.

Combining this proposition for the simplest $\sigma = \mathrm{id}$ with the natural extensions to greater root systems we come to the following corollary, essentially, sufficient to obtain all classical affine minimal NGT.

Corollary 2.7. *Let R' be a root subsystem of R such that the corresponding Dynkin diagram $\Gamma' \subset \Gamma$ is connected and contains B_3, C_3 or D_4 as a subdiagram,*

$$\varpi_{b'} = b(\upsilon_{b'})^{-1} \in \widehat{W}' \ \text{ for } \ \upsilon_{b'} = w_0' w_0^{b'},$$

subject to the conditions for $b' \in P'$ from the theorem (with $\sigma' = id$). We linearly extend the embedding $Q' \subset Q \subset \mathbb{R}^n$ to $P' \subset \mathbb{R}^n$.

(i) Let us assume that $b' \in P$ and that there exists $b \in P$ such that $b - b'$ is a linear combination of the fundamental weights ω_j for $\alpha_j \notin \Gamma'$, provided the conditions

$$\begin{aligned} (2.25) \qquad & b \in P_+^{(k')}, \ \ \overline{b} \overset{\mathbf{def}}{=\!=} -\upsilon_{b'}(b) \ \in \ P_+^{(\overline{k}')}, \\ & (b, \vartheta) \ \le \ 0 \ \ge \ (\overline{b}, \vartheta) \quad for \quad \vartheta \in R_+, \end{aligned}$$

where $k', \overline{k}'$ correspond to $b', \overline{b'}$. Then $\widehat{w} \overset{\mathbf{def}}{=\!=} b(\upsilon_{b'}(b))^{-1}$ represents a minimal NGT for $\widetilde{R}$.

(ii) If $b = b'$ here (the extension by zero), then $\overline{b} = -\varsigma'(b)$ for ς' defined for R'_+. In this case one must check that $(\varsigma'(b'), \alpha_m) \ge 0$ for any simple root α_m neighboring Γ' in Γ (two may occur for $E_{6,7,8}$). Also, if $\alpha_0 = [-\vartheta, 1] \in \widetilde{R}$ is connected inside the affine diagram $\widetilde{\Gamma}$ for $\widetilde{R}$ with one of the vertices of Γ' by a link, then the conditions $(b, \vartheta) \le 0 \ge (\varsigma'(b), \vartheta)$ must hold. Such $\widehat{w}$ represents a minimal NGT.

Proof. Let us demonstrate that the endpoints of $\widehat{w}$ coincide with those of $\varpi^{\cdot}_{b'}$ (and are unique). Representing $\widehat{w} = (\upsilon^{\cdot}_{b'})^{-1}(-\overline{b})$, we see that new *simple* nonaffine roots can appear only due to the set $\lambda(-\overline{b}) \subset \widetilde{R}$. If $b = b'$ then $\overline{b} = -\varsigma'(b)$ and they have to be among the neighbors α_m of Γ' in Γ such that $(\alpha_m, -\varsigma'(b')) > 0$. The analysis of $\alpha_0 \in \widetilde{R}_+$ is straightforward using (2.18). □

We note that the special case $b = b'$ is actually covered by Proposition 2.3. The *dots* in this corollary can be ignored for $b' \neq 0$. If $b' = 0$, i.e., the initial $\varpi_{b'}$ is non-affine, then b' can be extended only by zero due to the relations in terms of ϑ.

One can combine the transformations $\widehat{w} \mapsto \pi_r \widehat{w}$ and $\widehat{w} \mapsto \widehat{w}\pi_r$ for $\pi_r \in \Pi$ with the construction from the corollary; see (2.21). Generally, the resulting elements will be "new", i.e., not covered this corollary for any proper b' and their extensions b. Multiplication by π_r here may change the centralizer of b and result in significant changes of the $\sigma-$elements of the theorem.

Furthermore, using the above transformations and automorphisms of $\widetilde{\Gamma}$, one obtains all affine minimal NGT of classical types.

The next corollary is about applications of our construction to *nonaffine* minimal NGT.

Corollary 2.8. *Let ϖ be the element from part (ii) of Theorem 2.6 satisfying the assumptions there. We require the positivity of β; for instance, the setting of Proposition 2.3 is sufficient. Using the decomposition $\varpi = \pi_\varpi u_\varpi$ from Proposition 1.1, the element $u_\varpi \in W$ represents a nonaffine minimal NGT $\{\beta, \gamma, \alpha_{\overline{k}}\}$ under the following condition. For any end β' of the sequence $\lambda(u_\varpi)$, the root $[\beta', -(\beta', \overline{b})]$ must be from $\lambda(\varpi)$, equivalently, $(\sigma^\varsigma)^{-1}(\beta') > 0$. All classical nonaffine minimal NGT can be obtained as u_ϖ for $b = 0$ and $\sigma = id$, possibly, with further embedding into a greater root system via Corollary 2.7.*

Proof. The claim concerning the classical root systems can be checked by inspection (see [CS] and below). Concerning the general statement, let us begin with clarifying the structure of $\lambda(\varpi)$. If σ is known (trivial or relatively simple), then the calculation of $\lambda(u_\varpi)$ for $\varpi = \varpi^\sigma_b$ becomes sufficiently explicit. Let us use (2.19) in the following form:

$$\lambda(\varpi^\sigma_b) \; = \; \overline{b}\Big(R_+ \setminus \big(\dot{w}^{\overline{b}}_0(\lambda(\overline{\sigma}) \cup R^{\overline{b}}_{+\cdot})\big)\Big) \,\widetilde{\cup}\, \lambda(-\overline{b}). \tag{2.26}$$

Here the nonaffine roots can appear due to $\lambda(-\overline{b})$ and because there can exist $\alpha \in R_+$ satisfying $(\alpha, \overline{b}) = 0$ not from $R_{+\cdot}^{\overline{b}}$. Thus,

$$\begin{aligned} \lambda(u_\varpi) = \{\alpha \in R_+ \backslash \big(\dot{w}_0^{\overline{b}}(\lambda(\overline{\sigma}) \cup R_{+\cdot}^{\overline{b}}\big) \mid (\alpha, \overline{b}) = 0\} \\ \widetilde{\cup}\, \{\alpha \in R_+ \mid (\alpha, \overline{b}) < 0\}, \end{aligned} \tag{2.27}$$

where the set in the second line obviously does not intersect the first one.

Apart from the roots with $(\alpha, \overline{b}) = 0$, the description of $\lambda(\varpi)$ and $\lambda(u_\varpi)$ becomes simpler. Let us use the relation (2.5)

$$\overline{\sigma}\dot{w}_0^{\overline{b}} \;=\; \dot{w}_0^{b^\varsigma}(\sigma^\varsigma)^{-1}. \tag{2.28}$$

First, we describe the set of all $[\alpha, \nu_\alpha j]$ in $\lambda(\varpi)$ with $\alpha > 0$ such that $(\alpha, \overline{b}) = -q < 0$. The inequality for j is $0 \le \nu_\alpha j \le q$ if α is *not* from $\lambda((\sigma^\varsigma)^{-1})$. We use that $[\alpha, q]$ belongs to the first set in the union from (2.26). Otherwise, i.e., when $\alpha \in \lambda((\sigma^\varsigma)^{-1})$, the inequality is $0 \le \nu_\alpha j < q$. Summarizing,

$$\begin{aligned} \lambda(\varpi) \cap [\alpha, \mathbb{Z}_+] \;=\; \\ \begin{cases} \{[\alpha, \nu_\alpha j] \mid 0 \le \nu_\alpha j \le q\} \text{ if } \alpha \notin \lambda((\sigma^\varsigma)^{-1}), \\ \{[\alpha, \nu_\alpha j] \mid 0 \le \nu_\alpha j < q\} \text{ if } \alpha \in \lambda((\sigma^\varsigma)^{-1}). \end{cases} \end{aligned} \tag{2.29}$$

Second, let $\alpha > 0$ and $(\alpha, \overline{b}) = q > 0$. In contrast to the previous case, σ may lead to diminishing the corresponding part of $\lambda(\varpi)$:

$$\begin{aligned} \lambda(\varpi) \cap [-\alpha, \mathbb{Z}_+] \;=\; \\ \begin{cases} \{[-\alpha, \nu_\alpha j] \mid 0 < \nu_\alpha j < q\} \text{ if } \alpha \notin \lambda((\sigma^\varsigma)^{-1}), \\ \{[-\alpha, \nu_\alpha j] \mid 0 < \nu_\alpha j \le q\} \text{ if } \alpha \in \lambda((\sigma^\varsigma)^{-1}). \end{cases} \end{aligned} \tag{2.30}$$

Let us now verify the claim concerning $\lambda(u_\varpi)$. Recall that $\widetilde{\beta} = -w_0\sigma\dot{w}_0^b(\alpha_k) = [\beta, p]$ for $p = -(b, \alpha_k) = -(\overline{b}, \beta)$. Thus the relation we imposed on β' is satisfied for β. One can assume that $p > 0$, which excludes the case of zero $b, \overline{b}$.

In the reduced decomposition $\varpi = \pi_\varpi u_\varpi$ for $\varpi = \varpi_b^\sigma$, the set $\lambda(\pi_\varpi)$ does not contain nonaffine roots, so the first (and simple) nonaffine root in the λ–sequence of ϖ must be from $\lambda(u_\varpi)$. Thus, the condition that the beginning of the $\lambda(\varpi)$ is not movable (under the Coxeter transformations) is *equivalent* to the corresponding property of u_ϖ provided that $\alpha_0 \notin \lambda(\varpi)$.

Using the positivity of β from $\widetilde{\beta}$, we conclude that $\beta \in \lambda(u_\varpi)$. So does $\gamma = \beta + \alpha_{\overline{k}}$. By assumption, if $\beta' \neq \beta$ is an end of the sequence $\lambda(u_\varpi)$, then it can be "lifted" to the root $\widetilde{\beta}' = [\beta', p']$ from $\lambda(\varpi)$, where $p' = -(\overline{b}, \beta')$.

We will use the following interpretation of the (left) *ends* $\widetilde{\beta}'$ of a λ–sequence (see [C1]). There must be no decompositions $\widetilde{\gamma}' = \widetilde{\beta}' + \widetilde{\alpha}'$ in terms of positive roots (including the pure imaginary ones) where $\widetilde{\gamma}'$ belongs to a given λ–sequence and $\widetilde{\alpha}'$ does not. Also, there must be no decompositions $\widetilde{\beta}' = \widetilde{\alpha}' + \widetilde{\alpha}''$ with $\widetilde{\alpha}'$ and $\widetilde{\alpha}''$ from this λ–sequence. These conditions are necessary and sufficient.

The decomposition $\widetilde{\beta}' = [\beta', p'] = [\alpha', r'] + [\alpha'', r'']$ in terms of the roots from $\lambda(\varpi)$ with positive α' and α'' is impossible, since $\beta = \alpha' + \alpha''$ contradicts the assumption that β is a (left) end of $\lambda(u_\varpi)$. However, it can be in the form $\widetilde{\beta}' = [\alpha', r'] + [-\alpha'', r'']$ for positive α' ($r' \geq 0$) and α'' ($r'' > 0$). Then $\alpha' \in \lambda(u_\varpi)$ and $\alpha'' \notin \lambda(u_\varpi)$. Hence, $\alpha' = \beta + \alpha''$ and β cannot be an end of $\lambda(u_\varpi)$ since it must appear before α'.

Next, let us consider the case when $\lambda(\varpi) \ni [\gamma', q'] = \widetilde{\beta}' + [\alpha', r']$ for $[\alpha', r'] \notin \lambda(\varpi)$. If $\alpha' > 0$ then β' cannot be the end. If $\alpha' = -\alpha$ for $\alpha > 0$ and $\gamma' > 0$, then $\gamma' \in \lambda(u_\varpi)$ and $\beta = \gamma' + \alpha$. Therefore, $\alpha \notin \lambda(u_\varpi)$. One can assume here (and below) that $r' = 1$. The relation $\alpha \notin \lambda(u_\varpi) \not\ni [-\alpha, 1]$ results in $(\overline{b}, \alpha) = 0$. Therefore, $(\gamma, -\overline{b}) = (\beta, -\overline{b}) = p'$. It contradicts to $q' = p' + 1 > p'$. See (2.29); note that this formula gives the reformulation of the assumption concerning the ends β' of $\lambda(u_\varpi)$ we imposed.

The remaining case is when $\alpha' = -\alpha < 0, r' = 1$ and $\gamma' = -\gamma < 0$. Then $\widetilde{\beta}' + [\gamma, -1 - q'] = [\alpha, -1]$. If $\alpha \in \lambda(u_\varpi)$, the root β must appear in this sequence before α, which is impossible. Therefore, $(\overline{b}, \alpha) = 0$ and $[-\gamma, -(\gamma, -\overline{b}) + 1]$ belongs to $\lambda(\varpi)$; a contradiction. See (2.30) for detail. □

It is of interest to explore the procedure from Corollary 2.8 for obtaining nonaffine minimal NGT for the exceptional root systems; we have no claims so far concerning the completeness.

3. NGT OF TYPE $\boldsymbol{B}$

The root system $\widetilde{B}_n (n \geq 3)$ is the key. Due to our choice of ϑ (it is the maximal *short* root; the twisted case), the corresponding affine Dynkin graph $\widetilde{\Gamma}$, Γ extended by $\alpha_0 = [1, -\epsilon_1]$, is the one from the C–table of

[B] where all the arrows are reversed. Concerning the normalization of the inner product, our one is $(\epsilon_i, \epsilon_j) = 2\delta_{i,j}$ for the Kronecker delta in terms of the basis $\{\epsilon_j\}$ from [B],

The lattice Q is generated by $\{\epsilon_j\}$. The action of $\widetilde{W}$ in $\mathbb{R}^{n+1} = [z, \zeta]$ and $\mathbb{R}^n \in z = (z_1, \ldots, z_n)$ (see (1.3) and (1.4)for $\xi = 1$) is as follows:

$$
\begin{aligned}
s_0([z,\zeta]) &= [(-z_1, z_2, \ldots, z_n), \zeta - 2z_1],\\
\epsilon_j((z)) &= z + \epsilon_j \quad \text{for} \quad z = (z_1, \ldots, z_n).
\end{aligned}
$$

We will use the involution of $\widetilde{\Gamma}$ transposing $\alpha_0 = [-\epsilon_1, 1]$ and $\alpha_n = \epsilon_n$; it will be denoted by $\imath_B$. It coincides with the conjugation by $\pi_n \in \Pi$.

For $\widetilde{C}_n$, the lattice Q is generated by $\epsilon_i \pm \epsilon_j$ including $2\epsilon_i$. Also, $\alpha_0 = [-\vartheta, 1]$ for $\vartheta = \epsilon_1 + \epsilon_2$ and $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Accordingly,

$$
\begin{aligned}
&s_0([z,\zeta]) = [(-z_2, -z_1, z_3, \ldots, z_n), \zeta - z_1 - z_2],\\
&(\epsilon_i \pm \epsilon_j)((z)) = z + \epsilon_i \pm \epsilon_j \quad \text{for} \quad z = (z_1, \ldots, z_n).
\end{aligned}
$$

The affine Dynkin diagram for $\widetilde{C}_n$ is the one from [B] for B with all arrows reversed. Its involution, transposing $\alpha_0 = [-\epsilon_1 - \epsilon_2, 1]$ and $\alpha_1 = \epsilon_1 - \epsilon_2$ and fixing the other simple roots, will be denoted by $\imath_C$. In terms of ϵ_i, it sends $\epsilon_1 \mapsto -\epsilon_1$ and leaves all other ϵ_j unchanged.

The lattice P for C coincides with Q for B. Sometimes we will denote α_0, ϑ, s_0 of type C by $\alpha_0', \vartheta', s_0'$ (the same for the related objects) to avoid confusions with those defined for B. For instance, the element s_0 from B can be treated as an element from $\widehat{W} = W \ltimes P'$ defined for C. Namely, $\Pi' = \{\mathrm{id}, s_0\}$, i.e., $\pi_1' = s_0$. We see that s_0 induces $\imath_C$.

3.1. **Configurations of type *B*.** Let us begin with a simple typical example of minimal affine NGT of type B presented in Figure 1.

There are $n = 6$ lines there which intersect and also experience reflections in the two *mirrors*. The bottom one will be always made parallel to the x–axis, the top one makes the angle $\delta/2$ with this axis.

Here and further by a *line* we mean a piecewise linear zigzag line which is the result of reflections of the initial line in the *mirrors*. The latter will be referred to as the *bottom nonaffine* mirror and the *top affine* one.

Almost always we consider only the portion of such zigzag lines trapped between the vertical lines at the beginning and at the end of the graph.

Configurations are defined as sets of (zigzag) lines between a given pair of vertical lines (the beginning and the end) where the triple intersections and double reflections are not allowed.

The *initial* angles the lines make with the x–axis (counterclockwise) will be denoted by $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$; we simply use $1, 2, \ldots, n$ in the graphs. They are numbered from top to bottom; accordingly, the initial angle between line i and line j for $j > i$ is denoted $i - j$. We read the configuration from right to left, so its beginning is the extreme right vertical line.

Let us list and interpret the geometric features of configurations aiming at establishing connections with the algebraic theory of $\widetilde{B}_n$ and the corresponding $\widetilde{W}$.

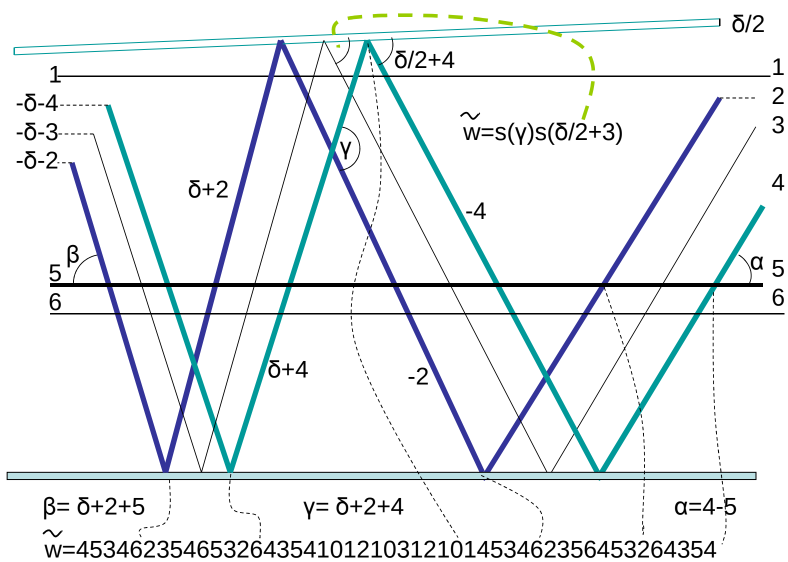

FIGURE 1. A basic affine NGT of type B

Geometric features of configurations.

(i) The construction of $\widetilde{w} \in \widetilde{W}$ from a given configuration is explained in the figure; see also [CS] and [C3]. More formally, we intersect the (zigzag) lines with the extreme left vertical line and read the

intersection points from top to bottom, forming the sequence of the *absolute* angles, which the lines make with the x–axis. This sequence can be uniquely represented as follows:

$$\delta(b_1, b_2, \ldots, b_n) + w(1, 2, \ldots, n) \text{ for proper } b_i \in \mathbb{Z} \text{ and } w \in W. \tag{3.1}$$

Then the element $\widetilde{w} \in \widetilde{W}$ (of type $\widetilde{B}$) associated with the configuration is defined as the product $\widetilde{w} = bw$. Here the vector $(b_1, \ldots, b_2)$, which equals $(0, -1, -1, -1, 0, 0)$ in the figure, is naturally identified with the weight $b = \sum_{i=1}^{n} b_i \epsilon_i \in Q$. Recall, that in the nonaffine theory of classical Weyl group of types B, C, D elements $w \in W$ are naturally identified with *permutations with signs*. For this particular configuration, $w = (1, -4, -3, -2, 5, 6)$.

Notice that the "unit" here is δ (not $\delta/2$ as in the interpretation of the affine roots); only integral multiples of δ appear in the angles. For instance, the vector of the absolute angles after the event s_0 is $\delta(1, 0, 0, \cdots) + (-1, 2, 3, \cdots)$. Thus the corresponding b equals $\epsilon_1 = \vartheta, w = s_\vartheta$, which matches the formula $s_0 = \vartheta s_\vartheta$.

As an exercise, check that $\widetilde{w}$ from the figure can be represented as a product of two pairwise commutative reflections $s_{[1,\epsilon_2+\epsilon_4]}$ and $s_{[1,\epsilon_3]}$.

(*ii*) The sequence of projections of the intersection points and the reflection points onto the x–axis gives the *reduced decomposition* of $\widetilde{w}$ corresponding to a given configuration. *We always assume that these projections are distinct.* Then their number equals the length $l(\widetilde{w})$. The simple reflections s_i $(0 \le i \le n)$ associated with the corresponding *simple events*, the intersections and the reflections, are determined on the basis of the *local line numbers* (always counted from top to bottom) at the moment of the event.

For disconnected events (corresponding to pairwise commutative s_i and s_j) we can of course change the order of the projections arbitrarily; we do it constantly in the figures.

Note that if "pseudo-lines" are allowed here, then all reduced decompositions of a given $\widehat{w}$ can be obtained in this way. Pseudo-lines are essentially the curves with one-to-one projections onto the x–axis that are allowed to intersect no greater than one time if no reflections are involved.

(*iii*) Next, the angles $\alpha + (\delta/2)j$ between the lines will be treated as the affine roots $[\alpha, j] \in \widetilde{R}$ (type $\widetilde{B}$). The angle is always calculated

counterclockwise and *before the event*, i.e., as the difference of the absolute incoming angles, the upper one minus the lower one. The events are intersections or reflections. The angles with the mirrors are taken for the reflections, namely, the absolute angles of the mirror are $\delta/2$ for the top one and 0 for the bottom one.

The angles correspond to *positive* affine roots, for instance, $\delta/2$ always occurs with a non-negative coefficient (even for the intersections). The collections of the corresponding angles considered from right to left constitute the λ–sequences $\lambda(\widetilde{w})$ of a given reduced decomposition of $\widetilde{w}$. If *pseudo-lines* are allowed instead of (straight) lines we consider, then all λ–sequences can be obtained in this way.

(iv) The action of $\widetilde{w}$ on the angles is *dual* to the affine action from (3.1). Practically, the image of ϵ_i considered as a root is the resulting angle of this line where index i is replaced by the *local* number of this line after the event (counted from top to bottom).

For instance, $\widetilde{w}(\epsilon_2 - \epsilon_5)$ in the figure under consideration is $\widetilde{w}(2-5) = (-\delta - 4) - 5 = -\delta - 4 - 5$ treated as the affine root $[-1, -\epsilon_4 - \epsilon_5]$. It is negative, so $2-5$ belongs to the list of the angles of this configuration.

Notice that the action of the lattice P (of type $\widetilde{B}$) requires an extension of the basic events by π_n transposing the affine Dynkin diagram $\widetilde{\Gamma}$. Recall that π_n is the only non-trivial element of Π. This event has no angle and does not contribute to the λ–sequences, although it of course transposes the line numbers and influences the angles afterwards.

Geometrically, let us assume that the mirrors are two generatrix lines of a circular 2–dimensional cone; then the configurations under consideration will belong to the one of the two halves of this cone. The reflection in the middle line between the mirrors in the *other half of the cone* naturally represents π_n. It transposes the mirrors and the corresponding lines between them; we denote it by $\imath_B$.

3.2. ***B*-positive minimal NGT.** We need to introduce some terminology.

A collection of *neighboring parallel lines* will be called a *bunch of lines*. The lines from a bunch are obtained from each other by (piecewise) parallel translations (adjusted to the mirrors).

Actually, by *parallel*, we mean here and below *combinatorially parallel*, i.e., the lines that "behave" as parallel and may intersect only due to the reflections (within the range where they are considered), *We always assume that any bunch is maximal possible* in a given configuration.

The lines from one bunch have the same numbers of top and bottom reflections. By horizontal, we mean the lines that are *parallel* (*combinatorially parallel*, to be exact) to the corresponding mirrors; then these numbers are zero. The t–*number* of a line is defined as the number of top reflections;

A natural generalization of the minimal NGT from Figure 1 is given in terms of the following data:

(a) the integers $u \geq 0, v \geq 1$ such that $m \stackrel{\mathbf{def}}{=\!=} n - u - v \geq 2$, which are the numbers of top and bottom horizontal parallel lines neighboring (the right ends and the left ends) the corresponding mirror;

(b) a decomposition $m = p_1 + p_2 + \ldots + p_r$ for positive integers p_j such that $p_r \geq 2$, which give the numbers of lines in the consecutive non-horizontal bunches (counted from top to bottom with respect to the right ends);

(c) a sequence of non-negative integers $0 \leq t_1 < t_2 < \ldots < t_r$, which are the t–numbers of the corresponding non-horizontal bunches;

(d) also, the number of the bottom reflections is assumed $t + 1$ for the bunches and t–numbers in (b);

The data from (a, b, c) determine the configuration uniquely due to assumption (d).

Geometrically, the horizontal bunches can be plotted arbitrarily close to the corresponding mirrors; the lines in one bunch can be plotted arbitrarily close to each other. In Figure 2, there are $1 + 1$ *horizontal* bunches near the top mirror and the bottom mirror (each with one line), namely, $\{1\}$ and $\{7\}$; then $t_1 = 0, t_2 = 1, t_3 = 2$ for the bunches $\{2\}, \{3, 4\}, \{5, 6\}$.

The t–number can be zero in our construction not only for the *horizontal* bunches. The first bunch of lines from (b) is allowed to have $t_1 = 0$. The presence of at least one *horizontal* bottom line ($v > 0$) is required. Also, the second bunch counted from the bottom, i.e., the first bunch from (b), must contain at least two lines ($t_r \geq 2$).

The first and the last lines from this bunch and the highest line in the bottom *horizontal* bunch will be exactly those responsible for producing the minimal NGT $\{\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\alpha}\}$ in the theorem below.

Note that the $\widetilde{w}$–elements corresponding to different configurations under consideration may have coinciding weights b. It occurs if and only if they have the same total number of lines with $t = 0$ due to a

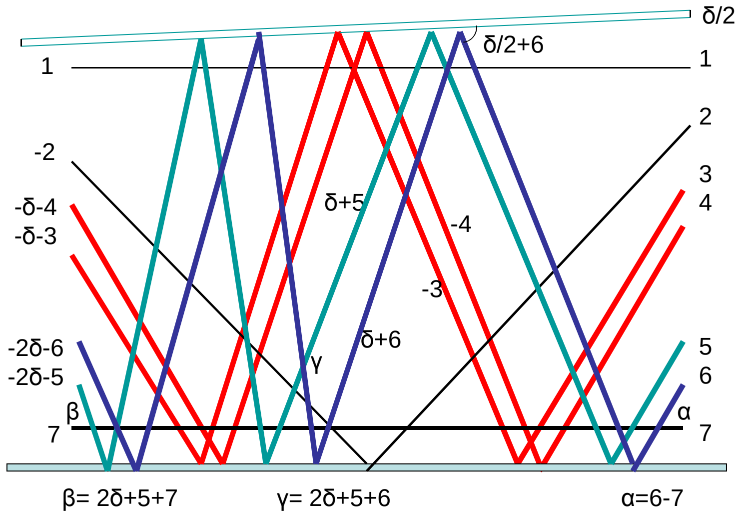

FIGURE 2. Basic affine NGT of type B

redistribution of lines between the top horizontal branch and the one with $t_1 = 0$.

This construction will be referred to as the *B−positive construction*; accordingly, such minimal NGT will be called *B−positive*. This name reflects the fact that the nonaffine component β of the root $\widetilde{\beta}$ is always positive in this construction. All minimal NGT with positive β can be obtained in this way; we come to the following theorem.

Theorem 3.1. *(i) Any minimal affine NGT $\widetilde{w} \in \widetilde{W}$ for the (twisted) root system $\widetilde{B}_n(n \geq 3)$ is either given by the B−positive construction in terms of (a, b, c, d) or can be obtain from a B−positive minimal NGT by applying the automorphism $\imath_B$ (transposing the top and the bottom mirrors). All such $\widetilde{w}$ are involutive.*

(ii) The B−positive $\widetilde{w}$ are covered by the construction of Corollary 2.7, (ii) for $k = \overline{k} = v+1$, where $v+1$ is the greatest line number in the bunch of lines for the last t_r (second from the bottom). The graph Γ' is obtained from Γ by removing the vertices $\alpha_1, \ldots, \alpha_u$, geometrically, by removing the top horizontal bunch (if present). □

3.3. **Proof.** We consider the configurations of the lines L_i discussed above and representing elements the $\widetilde{w} \in \widetilde{W}$ for $\widetilde{W}$ of type $\widetilde{B}$. The lines are numbered at the beginning (for the extreme right value of x). Each of L_i is characterized by the number of top reflections t_i and the number of bottom reflections b_i. More exactly, this numbers determines the type of L_i uniquely if $t_i \neq b_i$; otherwise, one needs to know which reflection (the top or the bottom one) occurs the first.

Let us begin with the following general observation.

Lemma 3.2. *Let the lines L_i and L_{i+1} be neighboring in the configuration corresponding to $\widetilde{w} \in \widetilde{W}$ of type $\widetilde{B}$.*

(a) If the first reflection of line L_i is in the bottom mirror and either $i = n$, or $b_i < b_{i+1}$ or $t_i < t_{i+1}$ for $i < n$, then the element s_i can be made the beginning of the reduced decomposition of $\widetilde{w}$.

(b) Similarly, s_i can be made the beginning of the reduced decomposition of $\widetilde{w}$ if L_{i+1} begins with the top reflection and either $i = 0$ or, in the case of $i > 0$, $t_i > t_{i+1}$ or $b_i > b_{i+1}$.

Proof. It suffices to check (a); also, the case $i = n$ is obvious. The geometric assumptions from (a) ensure that the angle $\epsilon_i - \epsilon_{i+1}$ occurs somewhere in such configuration. Indeed, the first reflection of line L_{i+1} (if any) can be only in the bottom mirror. Then lines $L - i$ and L_{i+1} can be made "parallel" (i.e., with the intersections only due to their reflections) until the first intersection. Since the lines have experienced the same number of the bottom and top reflections before the intersection, the angle between them has to be $\epsilon_i - \epsilon_{i+1}$. This angle corresponds to the simple root α_i; therefore it can be made the first upon a proper transformation of the configuration. ☐

Lemma 3.3. *The statement of Theorem 3.1 holds for $\widetilde{B}_3$.*

Proof. Using $\imath_B$ (the transposition of the two mirrors), one can assume that the first angle of the configuration representing a minimal NGT, $\{\widetilde{\beta}, \widetilde{\gamma} = \widetilde{\alpha} + \widetilde{\beta}, \widetilde{\alpha}\}$, is $\widetilde{\alpha} = \epsilon_2 - \epsilon_3$. Then the last one, $\widetilde{\beta}$, can be

(1) $m\delta + \epsilon_1 + \epsilon_3$, or (2) $m\delta + \epsilon_1 - \epsilon_2$ for $m \geq 0$, and, additionally,

(3) $m\delta - \epsilon_1 - \epsilon_2$, or (4) $m\delta - \epsilon_1 + \epsilon_3$ when $m > 0$.

Let us demonstrate that the last three choices are impossible. We will use Lemma 3.2.

First of all, the following holds:

a) L_2 reflects in the bottom mirror after the intersection with L_3,

b) the first reflection of L_1 may occur only in the bottom mirror,

c) $b_1 \le b_2$ for the numbers of the bottom reflections of L_1 and L_2,
d) the first reflection (if any) of line L_3 can be only in the top mirror.

Furthermore, a simple check gives that the angles between L_1 and L_2 will be always in the form $m\delta + \epsilon_2 \pm \epsilon_1$; this excludes (2) and (3).

A more algebraic verification is as follows. If the angle from (2) for $m > 0$ appears in the configuration, then so does $\epsilon_1 - \epsilon_2$. The latter represents a simple root and can be made the first one, which contradicts the minimality of NGT. Similarly, for the angle from (3), $\delta - \epsilon_1 - \epsilon_2$ is an angle too; it results in a contradiction too.

As for (4), line L_3 intersects L_1 when it goes down (after the corresponding top reflection) or up (after the corresponding bottom reflection). In either case, the sign of ϵ_1 in the intersection angle is always plus, so (4) is impossible.

Thus, (1) is the only option for $\widetilde{\beta}$. Let us now check that line L_3 is actually horizontal (i.e., does not reflect). We claim that if it reflects in the mirrors then its last reflection can be made the last event of the configuration, which contradicts the minimality of the NGT under consideration. Figure 3 demonstrates this claim; the thick arc there shows the reflection points that can be transposed in this configuration.

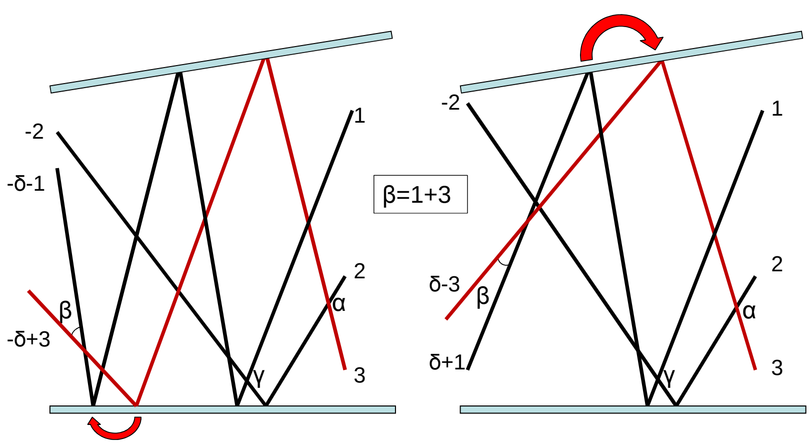

FIGURE 3. Line 3 must be horizontal

This conclude the verification of the lemma. □

Let us apply Lemma 3.3 to the three lines forming a minimal NGT $\{\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\alpha}\}$ for $\widetilde{B}_4$. Then the forth line can be one of the following:
1) horizontal near the bottom or near the top (the last or the first);
2) the one between the two non-horizontal parallel lines from the triple,
3) below these two non-horizontal lines and of type $b - t = 1$ with a b–number smaller than that for these two.

Generally, for a minimal NGT configuration of type $\widetilde{B}_n$, any new line can be either added to an existing *bunch of lines* or can "begin" a new bunch subject to the inequalities from Section 3.2. It includes the horizontal bunches near the bottom or near the top. We use Lemma 3.2. The theorem is proven.

4. Types $\boldsymbol{C - D}$

Figure 4 gives an example of a minimal affine NGT of type C constructed using a *parity correction* from a minimal NGT of type B. It also illustrates the construction from Theorem 2.6, including a geometric interpretation of b and σ.

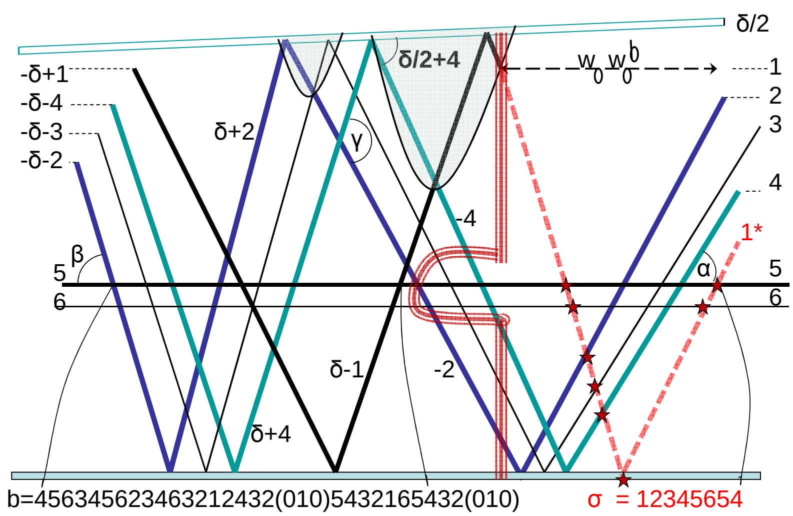

FIGURE 4. NGT of type C-D: b and σ

Let us use this graph to demonstrate the changes in the $\widetilde{C},\widetilde{D}$–cases versus the planar interpretation for $\widetilde{B}$. We will use prime ($\widetilde{W}'$, $\widehat{W}'$ and so on) for the objects considered for $\widetilde{C},\widetilde{D}$.

4.1. **Main modifications.** The general way of constructing the reduced decompositions in terms of the intersection and reflection points remains essentially the same. The elements $\widetilde{w}\in\widetilde{W}$ of type $\widetilde{B}$ always belong to $\widehat{W}'$ of type $\widetilde{C}$ or $\widetilde{D}$ (generally, not to $\widetilde{W}'$); the element s_0 for B is naturally interpreted as the generator of Π' for C or D. The following is necessary and sufficient for the inclusion $\widetilde{w}\in\widetilde{W}'$.

One needs to check that the total number of top reflections is even (for $\widetilde{C}_n$ and $\widetilde{D}_n$) and the total number of bottom reflections is even in the case of $\widetilde{D}_n (n\geq 4)$. Then we can transform such reduced decomposition to make it from $\widetilde{W}'$, i.e., in terms of new simple events $s_0'=s_0s_1s_0$ and $s_n'=s_ns_{n-1}s_n$; the latter is needed for D.

In Figure 4, if the dashed line to 1^* is added to line 1, then the corresponding word becomes of type $\widetilde{D}_6$ (from the corresponding $\widetilde{W}$). If this dashed line is disregarded here, then the corresponding configuration is of type $\widetilde{C}$ but not of type $\widetilde{D}$. If we disregard the dashed line completely and, moreover, remove the top-right reflection of line 1, then the resulting word is neither of type $\widetilde{C}$ nor of type $\widetilde{D}$. Let us discuss this example and related features of our construction in more detail.

First, let us begin with the element corresponding to the configuration where we disregard the portion of line 1 *before* (to the right of) the top-right reflection. Graphically, the dashed line is disregarded. We will denote it by $\widetilde{w}$. Adding this top reflection to $\widetilde{w}$ gives an example of the *top-right parity correction* of $\widetilde{w}$. Algebraically, $\widetilde{w}\mapsto\widetilde{w}'\stackrel{\mathbf{def}}{=\!=}\widetilde{w}s_0$. It makes the number of the top reflections even, so $\widetilde{w}'$ can be expressed in terms of s_0' instead of s_0 and becomes a word of type $\widetilde{C}$.

Second, let $\widetilde{w}''$ be the graph for $\widetilde{w}'$ extended by the dashed line ending at $1^\star$; then $\widetilde{w}''$ is of type D.

Actually, the simplest way of transforming the $\widetilde{w}'$ to a word of type D is via the *bottom parity correction* (right or left), i.e., using line 6. Algebraically, it is the transformation $\widetilde{w}'\mapsto\widetilde{w}'s_n=s_n\widetilde{w}'$.

Note that the *top-left parity correction* of $\widetilde{w}$, that is $s_0\widetilde{w}$, is different from the *top-right parity correction* $\widetilde{w}'=\widetilde{w}s_0$. Generally, the right

and left parity corrections coincide only if they are performed on the same horizontal line. Line 6 (used for the bottom-right correction) is *horizontal*; line 1 is not.

Concerning the interpretation of the angles as roots and related matters, there are the following modifications versus the $\widetilde{B}$–case.

(i) The angles for the bottom reflections must be multiplied by 2 for $\widetilde{C}$. The angle of $s_0' = s_0 s_1 s_0$ or $s_n' = s_n s_{n-1} s_n$, presented in terms of s_0, s_n for $\widetilde{B}$, is the middle one (from the three angles involved in this event).

(ii) The angles $j\delta + p \pm q$, including $j\delta + 2p$ in the $\widetilde{C}$–case, are transformed to the affine roots $[\,\epsilon_p \pm \epsilon_q\,,\, j\,]$; so this interpretation is different from the $\widetilde{B}$–case, where the "unit" was $\delta/2$. The graphic description of the action of $\widetilde{W}$ on the roots remains unchanged; we read the angles after the event, replacing their original numbers by the local ones.

In the figure under consideration, the angles of the two $\widetilde{D}$–type top events (marked) are correspondingly $\delta-1+4 = [\epsilon_1-\epsilon_4, 1]$ and $\delta+3+2 = [\epsilon_2+\epsilon_3, 1]$.

(iii) The interpretation of the sequence of the *absolute angles* (with the x–axis) at the end of the configuration as a representation of $\widetilde{w}$ in the form bw remains unchanged versus the $\widetilde{B}$–case.

Recall that we consider the sequence of absolute angles (counted from top to bottom) as a vector

(4.1) $\delta(b_1, b_2, \ldots, b_n) + w(1, 2, \ldots, n)$ for proper $b_i \in \mathbb{Z}$ and $w \in W$.

Then $\widetilde{w} = bw$, where we identify $b = (b_1, \ldots, b_2)$ with $\sum_{i=1}^n b_i \epsilon_i \in Q$. See (3.1). We continue using the notation from [B].

For instance, $\alpha_0 = [-\vartheta, 1]$, where $\vartheta = \epsilon_1+\epsilon_2$ for both, $\widetilde{C}$ and $\widetilde{D}$. The angle of $s_0' = s_0 s_1 s_0$ is $\delta-1-2 = [-\epsilon_1-\epsilon_2, 1]$. The vector of the absolute angles after this event is $\delta(1,1)+(-2,-1)$. Thus $b = \epsilon_1+\epsilon_2 = \vartheta, w = s_\vartheta$ and $s_0' = \vartheta s_\vartheta$.

Note that the lattice Q becomes smaller versus that for $\widetilde{B}$ (it is the same one for $\widetilde{C}$ and $\widetilde{D}$). Namely, it contains $b = \sum_{i=1}^n b_i \epsilon_i$ with integral b_i only for even $\sum_{i=1}^n b_i$.

For instance, $\alpha_0 = [-\vartheta, 1]$, where $\vartheta = \epsilon_1+\epsilon_2$ for both, $\widetilde{C}$ and $\widetilde{D}$. The angle of $s_0' = s_0 s_1 s_0$ is $\delta-1-2 = [-\epsilon_1-\epsilon_2, 1]$. The vector of the absolute

angles after this event is $\delta(1,1)+(-2,-1)$. Thus $b=\epsilon_1+\epsilon_2=\vartheta$ and $w=s_\vartheta$, which matches the relation $s'_0=\vartheta s_\vartheta$.

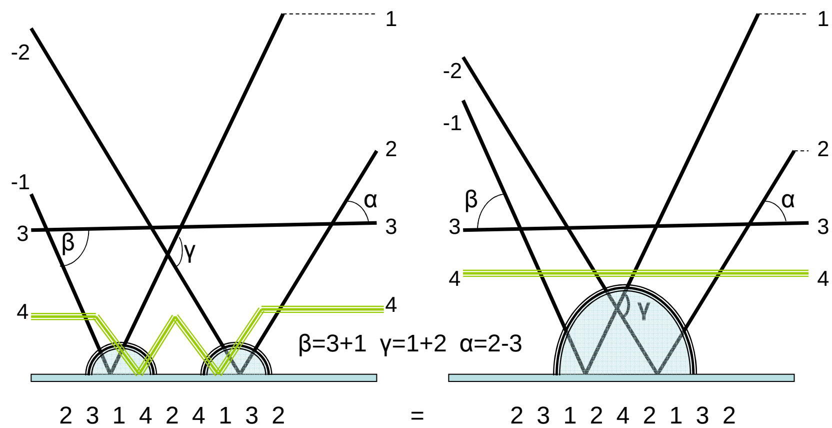

FIGURE 5. Type D, breaking the line

Concerning s'_n or s'_0, there is a special procedure for dealing with the graphs when the other lines are allowed to intersect (the area of) the corresponding triple event. It is called *breaking the line* and is directly related to the parity corrections. It is necessary for collecting the triples corresponding to s'_n or s'_0 in a given B–word (assuming that the latter satisfies the corresponding parity condition).

Figure 5 reproduces the graph from [CS], which demonstrates the procedure *breaking the line* and presents the simplest nonaffine minimal NGT of type D_4. Here 4 in the reduced decompositions stays for the simple (nonaffine) reflection s'_4 for D_4, corresponding to the triple product $s_4s_3s_4$ from the viewpoint of nonaffine B or C. This graph proves the Coxeter relation $s'_ns_{n-2}s'_n=s_{n-2}s'_ns_{n-2}$ $(424=242)$.

4.2. **The classification theorem.** Theorem 2.6, generally, provides adequate algebraic tools for the classical minimal NGT. As a matter of fact, its reduced version, Corollary 2.7, is sufficient to obtain all classical minimal NGT if one uses the automorphisms of the affine

Dynkin diagram and the passage from Q to P. This corollary can be used to justify *algebraically* that our geometric constructions (in the classical cases) really result in minimal NGT. However, concerning the completeness, we rely on the planar interpretation. The algebraic approach based on Theorem 2.6 is expected to be useful in the theory of minimal NGT for the exceptional root systems.

The discussion above leads to the following theorem. Slightly abusing the terminology, by a *mirror of type* D, we mean the top mirror for $\widetilde{C}$ or an either one for $\widetilde{D}$ (i.e., when the corresponding end of the affine Dynkin diagram is of type D).

Theorem 4.1. *(i) The group $\widehat{W}'$ of type $\widetilde{C}$ can be naturally identified with the group $\widetilde{W}$ of type $\widetilde{B}$, where $s_0 \in \widetilde{W}$ is interpreted as the generator of $\Pi' \subset \widehat{W}' = \widetilde{W}$. Accordingly, the configurations for the elements from $\widetilde{W}$ become configurations for $\widehat{W}'$ and an arbitrary element of $\widehat{W}'$ can be obtained from a configuration of type $\widetilde{B}$.*

(ii) The resulting elements belong to $\widetilde{W}'$ of type $\widetilde{C}$ if and only if the number of the top reflections of the corresponding configuration is even. Furthermore, they belong to $\widetilde{W}'$ of type $\widetilde{D}$ if the number of bottom reflections is even too. Given $\widetilde{w}' \in \widetilde{W}'$, it can be obtained from a $\widetilde{B}$–type configuration where all s_0 and s_n are included in the events $s_0' = s_0 s_1 s_0$ and $s_n' = s_n s_{n-1} s_n$.

(iii) All minimal NGT from $\widetilde{W}'$ of type $\widetilde{C}$ or $\widetilde{D}$ come from minimal NGT of type $\widetilde{B}$ satisfying (ii). However, the latter may become non-minimal in $\widetilde{W}'$; it occurs only if the horizontal line involved in the NGT is a unique one in its (horizontal) bunch and if the corresponding mirror is of type D.

Proof. The statements from (i, ii) have been already discussed. Concerning $\widetilde{C}$, we move all s_0 to the beginning (or the end) of a given reduced decomposition of $\widetilde{w} \in \widetilde{W}$, replacing $s_0 s_1 s_0$ by s_0' when necessary. It will give a word from $\widetilde{W}'$ possibly multiplied by s_0 on the right (or on the left).

As for $\widetilde{D}$, we can use that the group $\widetilde{W}$ for $\widetilde{B}$ can be naturally identified with the extension of $\widetilde{W}'$ of type $\widetilde{B}$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the elements s_0, s_n (pairwise commutative) treated as outer automorphisms of the corresponding affine Dynkin diagram. The element s_0 is

from $\widehat{W}'$, the element s_n is not; both are of zero length by definition. One can move all such elements in a given reduced $\widetilde{B}$–decomposition to its beginning or to its end. The elements s'_0 and s'_n may be produced during this process. The top (or bottom) parity corrections are needed if the elements s_0 (or s_n) do not cancel each other.

Algebraically, (ii) means that for any given $\widetilde{w}' \in \widetilde{W}'$, its reduced decompositions with the minimal possible numbers of s'_0 and s'_n remain reduced in $\widetilde{W}$, where s'_0 and s'_n are expressed in terms of s_0 and s_n. The geometric approach guarantees that at least one such reduced decomposition exists. The construction $\widetilde{w} \mapsto \widetilde{w}'$ (subject to the parity conditions) consists of moving all s_0, s_n to the beginning (or to the end) of a given reduced decomposition of $\widetilde{w}$.

Concerning (iii), let us use the right graph in Figure 5 to demonstrate what happens in the beginning of the configuration if there is only one line parallel to the corresponding mirror. If (the lowest) line 4 is removed, then the corresponding reduced decomposition reads $s_2 s_1 s'_3 s_1 s_2$. Using the Coxeter D–relation, it can be transformed to $s_2 s'_3 s_1 s'_3 s_2 = s'_3 s_2 s_1 s_2 s'_3$. Thus, the beginning of the resulting reduce decomposition is movable (not unique) and such B–minimal NGT will not remain D–minimal.

This example is actually a general one; it is sufficient to manage arbitrary $\widetilde{C}_n$ and $\widetilde{D}_n$ if the horizontal line involved in the NGT is near the mirror of type D and is the only one in its horizontal bunch. Then the right end of the resulting λ–sequence becomes movable upon the switch to $\widetilde{C}_n$ or $\widetilde{D}_n$ in this case (and only in such case). $\square$

For instance, let us consider the configuration from Figure 4 including the dashed line to $1^\star$ and excluding line 6. It represents a minimal NGT for $\widetilde{B}_5$, which *will not* remain minimal upon its recalculation to $\widetilde{D}_5$ (which is possible because the numbers of top and bottom reflections are both even). It is analogous to the constraint "at least two horizontal bottom lines" from [CS] in the nonaffine D–case.

The following theorem is an explicit form of claim (iii), reformulated in terms of the parity corrections.

Theorem 4.2. *(i) An arbitrary minimal NGT from* $\widetilde{W}$ *of type* $\widetilde{C}_n (n \geq 3)$ *can be obtained as* $\widetilde{w}'$, $\widetilde{w}''$ *or* $\widetilde{w}^*$ *as follows.*

Let $\widetilde{w}$ *be a B–positive minimal NGT of type* $\widetilde{B}_n (n \geq 3)$. *If no top parity correction is needed for* $\widetilde{w}$ *(i.e., the total number of top reflections*

in $\widetilde{w}'$ is even), then $\widetilde{w}' \stackrel{\mathbf{def}}{=\!=} \widetilde{w}$, (considered as $\widetilde{C}$–words) and, moreover, $\widetilde{w}'' = \imath_C(\widetilde{w}) = s_0\widetilde{w}s_0$ are minimal NGT of type $\widetilde{C}_n$. The element $\widetilde{w}^*$ is minimal NGT if it has even number of the top reflections and the initial $\widetilde{w}$ has at least two bottom horizontal lines; $\widetilde{w}^*$ is involutive and coincides with $\imath_C(\widetilde{w}^*)$.

Otherwise, if the parity corrections are needed, $\widetilde{w}' \stackrel{\mathbf{def}}{=\!=} \widetilde{w}s_0$ (the results of the top-right parity correction) is a minimal NGT of $\widetilde{C}_n$–type. For the top-left correction, $\widetilde{w}'' \stackrel{\mathbf{def}}{=\!=} s_0\widetilde{w} = \imath_C(\widetilde{w}') = (\widetilde{w}')^{-1}$, so such element can be represented as $\widetilde{u}'$ for certain B–positive $\widetilde{u}$. The element $\widetilde{w}^*$ defined now as $\imath_B(\widetilde{w})s_0 = s_0\imath_B(\widetilde{w})$, is a minimal NGT subject to the same "2 line constraint" as above.

(ii) Minimal NGT of type D are given in terms of the C–words $\widetilde{w}', \widetilde{w}'', \widetilde{w}^*$ from part (i) as follows. If no bottom parity correction is needed, i.e., the total number of the bottom reflections in the initial $\widetilde{w}$ (the cases of $\widetilde{w}'$ and $\widetilde{w}''$) or $\imath_B(\widetilde{w})$ (the case of $\widetilde{w}^*$) is even, then each of these elements is a minimal NGT of type $\widetilde{D}_n$. We require here the existence of at least two horizontal lines near the bottom for $\widetilde{w}$ in the cases of $\widetilde{w}'$ and $\widetilde{w}''$.

Otherwise, if the total number of the bottom reflections for $\widetilde{w}'$, $\widetilde{w}''$ or for $\widetilde{w}^*$ is odd, then the elements $s_n\widetilde{w}' = \widetilde{w}'s_n$ and $s_n\widetilde{w}'' = \widetilde{w}''s_n$, as well as the elements $\widetilde{w}^*s_n$ and $s_n\widetilde{w}^* = (\widetilde{w}^*s_n)^{-1}$ are minimal NGT of D–type. We impose the same constraint as above in the cases of $\widetilde{w}', \widetilde{w}''$, namely, the configuration for $\widetilde{w}$ is supposed to contain at least two horizontal lines near the bottom. □

We note that the operation $\widetilde{w} \mapsto s_0\widetilde{w}s_0$ for B–positive $\widetilde{w}$ is trivial if $\widetilde{w}$ contains at least one *top horizontal* line. It is obvious geometrically that the configuration for $s_0\widetilde{w}s_0$ can be transformed to ensure the cancelation of such two s_0 if they are "performed" on the same top horizontal line. Similarly, the transformation $\widetilde{w} \mapsto s_n\widetilde{w}s_n$ is not needed because it is always trivial; the bottom line of B–positive $\widetilde{w}$ is always *horizontal*.

4.3. **Algebraic aspects.** Let us now discuss Figure 4 from the viewpoint of "algebraic" Theorem 2.6 and Corollary 2.7. The C–elements $\widetilde{w}'$ and $\widetilde{w}''$ (the latter is $\widetilde{w}'$ extended by the dashed line to $1^\star$) have coinciding weights, so they must correspond to different σ. This weight is $b = -(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$ in the notation from [B]; the portion of

the graph after (to the left of) the solid thick vertical "line" gives its reduced decomposition. The "enclave" there excludes the intersections with lines 5 and 6 near this thick line, which simply means that we are supposed to make these two lines "very" close to the bottom to make the decomposition of b right.

We note that $l(\widetilde{w}')=l(b)+l(\upsilon_b^\sigma)$ for $\widetilde{w}'=b\upsilon_b^\sigma$ from the figure, which is generally not the case. It occurs if and only if $(b,\alpha)\le 0$ for all $\alpha\in R_+$.

For the configuration for tw'' (that includes the dashed line to $1^\star$), σ is identical, i.e., $(\upsilon_b^\sigma)^{-1}=(w_0w_0^b)^{-1}$ here. The element $w_0\dot{w}_0^b$ (recall that $w_0=-1$) is given by the portion of this configuration before (to the right of) the thick "line". It is shown as $w_0w_0^b$ in this picture. The *dot*–marks can be omitted here and below because $p>0$ and b is *not* dominant, namely, $p=-(b,\alpha_4)=-(b,\epsilon_4-\epsilon_5)=1$.

The example of $\widetilde{w}'=\varpi_b^\sigma$ (with 1 instead of $1^\star$), requires using a non-trivial σ (see the graph), resulting in "deleting" the dashed line. Its reduced decomposition is 12345654 (where we put i instead of s_1).

From the viewpoint of Corollary 2.7, we consider the extension from $\widetilde{R}'=\widetilde{B}_5$ to $\widetilde{R}=\widetilde{C}_6$, where $b'=-(\epsilon_2+\epsilon_3+\epsilon_4)$ is extended to $b=b'-\epsilon_1$. Then $\widetilde{w}'=bw_0'w_0^{b'}$, where $w_0'w_0^{b'}$ is calculated in $\widetilde{W}'$ for $\widetilde{B}_5$. It gives exactly $b(w_0\sigma w_0^b)^{-1}$.

Comment. Note, that the corresponding ϖ_b^σ is not a minimal NGT when considered as a word of type B. It is obvious from the picture. Algebraically, $\sigma(\vartheta)=\vartheta=\epsilon_1$ in B_6 and the corresponding condition from (2.18), $(b,\vartheta)\le 1$, fails. Recall, that our form is normalized by the condition $(\vartheta,\vartheta)=2$, so $(b,\vartheta)=2$.

When doing the same check for ϖ treated as a C–word, we use that ϑ becomes $\epsilon_1+\epsilon_2$ and our form coincides with the standard one (such that $\{\epsilon_j\}$ are orthonormal). Then $(b,\vartheta)=0\le 0$, which guarantees that the outcome is a minimal NGT. $\square$

The inversion. We will begin with an analysis of the inversion of the minimal NGT constructed above. Continuing the discussion of Theorem 2.6, let us consider $\widetilde{w}^\sharp \stackrel{\mathbf{def}}{=\!=} (\varpi_b^\sigma)^{-1}$ equals $\varpi_{\overline{b}}^{\overline{\sigma}}$, where $\overline{b}=\sigma(\varsigma(b))=\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_3$. Recall that $\widetilde{w}'=\varpi_b^\sigma$; see Figure 4. Graphically, $\widetilde{w}^\sharp$ describes the *top-left correction* of the element $\widetilde{w}$ above.

Figure 6 shows the configuration associated with $\widetilde{w}^\sharp=\overline{b}\overline{w}$; it corresponds to the case when the dashed line is ignored. Geometrically, it is

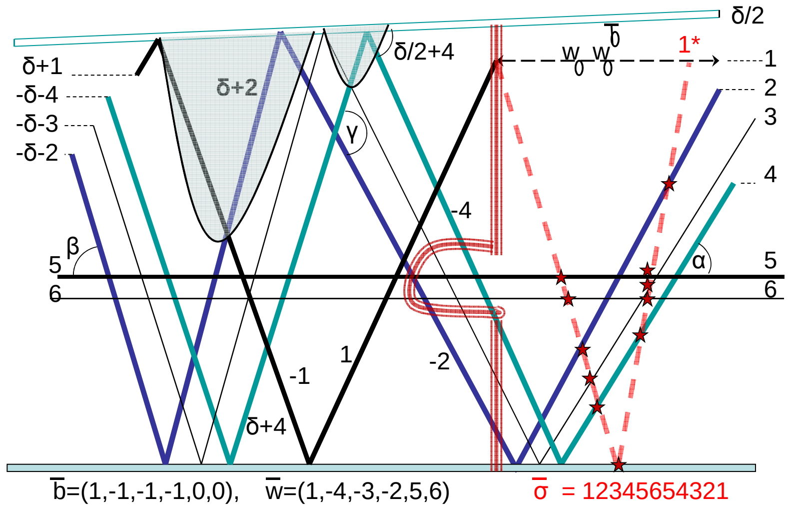

FIGURE 6. Type C: top-left correction

quite obvious that it represents a minimal NGT. Let us interpret this fact algebraically via Theorem 2.6.

The weights b and $\overline{b}$ belong to the same set $Q \cap P_+^{(4)}$ (have non-negative inner products with the same α_4), i.e., $\overline{k} = k = 4$. The condition (a') from (2.8) (which guarantees that α_k is a non-movable first root in $\lambda(\varpi_b^\sigma)$ holds. Moreover, (a''') from (2.13) holds. Namely, $\sigma(R_+^b) \supset R_+^{\overline{b}}$. Indeed,

$$\sigma(\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6) = (\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \supset (\alpha_2, \alpha_3, \alpha_5, \alpha_6).$$

For the sake of completeness, let us mention that $\sigma(\alpha_4) = -\epsilon_1 - \epsilon_5$.

The element $\overline{\sigma}$ equals 12345654321; so it is non-trivial and does not coincides with σ^{-1}. In Figure 6, $\overline{\sigma}$ is represented by the portion of the configuration before (to the right of) the vertical line where only the events involving the dashed line to 1^* are taken.

The element $\overline{b}$ is shown by the portion after (to the left of) the vertical line. Notice that the events for s_0' are combined in a way different from that in Figure 4. The element $w_0 w_0^{\overline{b}}$ corresponds to the portion of the configuration taken before the vertical line and where the dashed line to $1*$ is included; it is shown in the picture.

From the viewpoint of Corollary 2.7, we consider the extension from $\widetilde{R}'=\widetilde{B}_5$ to $\widetilde{R}=\widetilde{C}_6$, where $b'=-(\epsilon_2+\epsilon_3+\epsilon_4)$ is extended to $\overline{b}=b'+\epsilon_1$. Then $\widetilde{w}^\sharp=\overline{b}w'_0w_0^{b'}$, where $w'_0w_0^{b'}$ is calculated in $\widetilde{W}'$ for $\widetilde{B}_5$.

The condition (b''') from (2.13) is not satisfied for $\overline{b}$. However, (b'') from (2.11) holds:

$$\overline{\sigma}(R_+^{\overline{b}})\cup\lambda(\overline{\sigma}^{-1})\supset R_+^b,$$

where the root $\alpha_1\in R_+^b$, missing in $\overline{\sigma}(R_+^{\overline{b}})$, comes from $\lambda(\overline{\sigma}^{-1})$.

We note that the total word in Figure 6, *including the dashed line*, is *non-reduced*. It represents the element $\widetilde{w}^\flat \stackrel{\mathbf{def}}{=\!=} \varpi_{\overline{b}}$ with the trivial σ. This element is a minimal NGT of type $\widetilde{C}$ and also of type $\widetilde{D}$. The corresponding reduced word can be obtained by transforming line 1 (including the dashed portion) into a horizontal one; its two bottom reflections annihilate each other. The resulting word for $\widetilde{w}^\flat$ is a result of the *top parity correction* of a B–positive minimal NGT in a *horizontal* line, namely, line 1 upon making it horizontal.

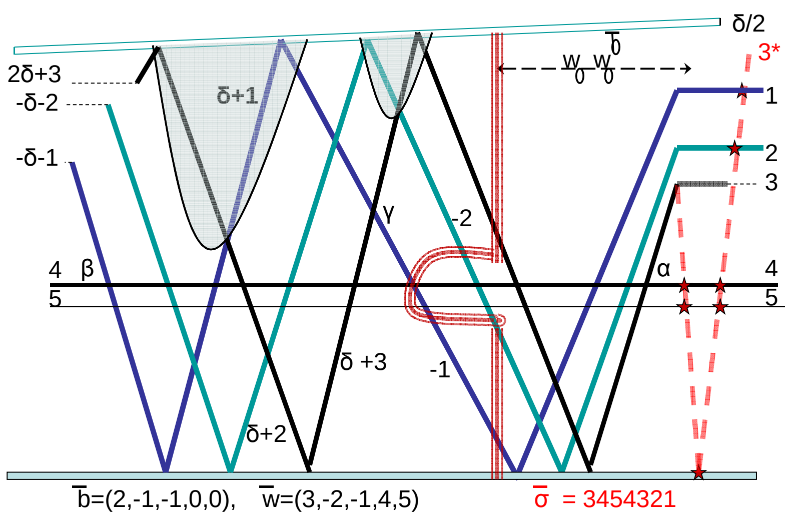

FIGURE 7. Type C: non-trivial σ

More on the parity corrections. The most involved ones are those performed on a line that is already involved in the *affine* reflections in the initial minimal NGT. Corollary 2.7 covers such parity corrections only if we apply it for a complete $\widehat{W}$ (with P instead of Q) with further reduction to $\widetilde{W}$ (the division by π_r). See (2.21).

A typical example of this situation is as follows. Let $\widetilde{w}' = b'w'$ for $b' = (-1,-1,-1,0,0)$ and $w' = (-3,-2,-1,0,0)$. In terms of the data (a,b,c), this B–positive element can be described as follows. First, $u=0$ (there is no top horizontal bunch of lines). Second, there is only one bunch from (b) with $t_1 = 1$, which contains 3 lines. Third, the bottom horizontal bunch contains 2 lines ($v=2$). The corresponding element $\widetilde{w}$ is of type $\widetilde{B}_5$ but not of type $\widetilde{C}_5$, since it involves 3 top reflections.

Let us perform the top-right correction. The resulting element $\widetilde{w}$ is of type $\widetilde{C}$ (but not of type $\widetilde{D}$). One has:

$$\widetilde{w} = bw \;\text{ for }\; b = (-1,-1,-2,0,0) \;\text{ and }\; w = (-3,-2,1,4,5).$$

The element w here is not of type $w_0 w_0^c$ (including the modifications from Corollary 2.7). See (2.24) for the algebraic meaning of the parity correction we used; the corresponding weight there is denoted by b''. We omit the graph of $\widetilde{w}$. Instead, we will focus on $\widetilde{w}^* = \widetilde{w}^{-1}$.

Figure 7 shows the configuration for $\widetilde{w}^*$; the dashed line must be dropped:

$$\widetilde{w}^* = \overline{b}\,\overline{w} \;\text{ for }\; \overline{b} = (2,-1,-1,0,0) \;\text{ and }\; \overline{w} = (3,-2,1,4,5).$$

This element is a result of the top-left correction of the initial element $\widetilde{w}'$. The expression for $\overline{\sigma}$ is provided in the figure. See (2.24); the corresponding weight there is b'.

The algebraic mechanism ensuring the absence of α_0 in $\lambda(\widetilde{w})$ and $\lambda(\widetilde{w}^*)$ is of some interest. The inner product $(\overline{b},\vartheta') = 1$ is critical for the second condition from formula (2.18); here $\vartheta' = \epsilon_1 + \epsilon_2$. So we need to check that $\overline{\sigma} w_0^{\overline{b}}(\vartheta') > 0$. One has:

$$w_0^{\overline{b}}(\vartheta') = \epsilon_1 + \epsilon_3,\; \overline{\sigma}(\epsilon_1 + \epsilon_3) = \epsilon_2 - \epsilon_3 > 0.$$

If the dashed line is added to $\widetilde{w}^*$, then the corresponding element $\widetilde{w}^\sharp$ is a result of the two simultaneous top parity corrections of the B–word $\widetilde{w}''$ constructed for $b'' = (0,-1,-1,0,0)$ and $w'' = (-1,-3,-2,4,5)$. See Figure 7: $\widetilde{w}^\sharp = s_0 \widetilde{w}'' s_0$. The corresponding σ–factor is trivial for $\widetilde{w}^\sharp$, so it is of the simplest type from the viewpoint of Theorem 2.6.

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